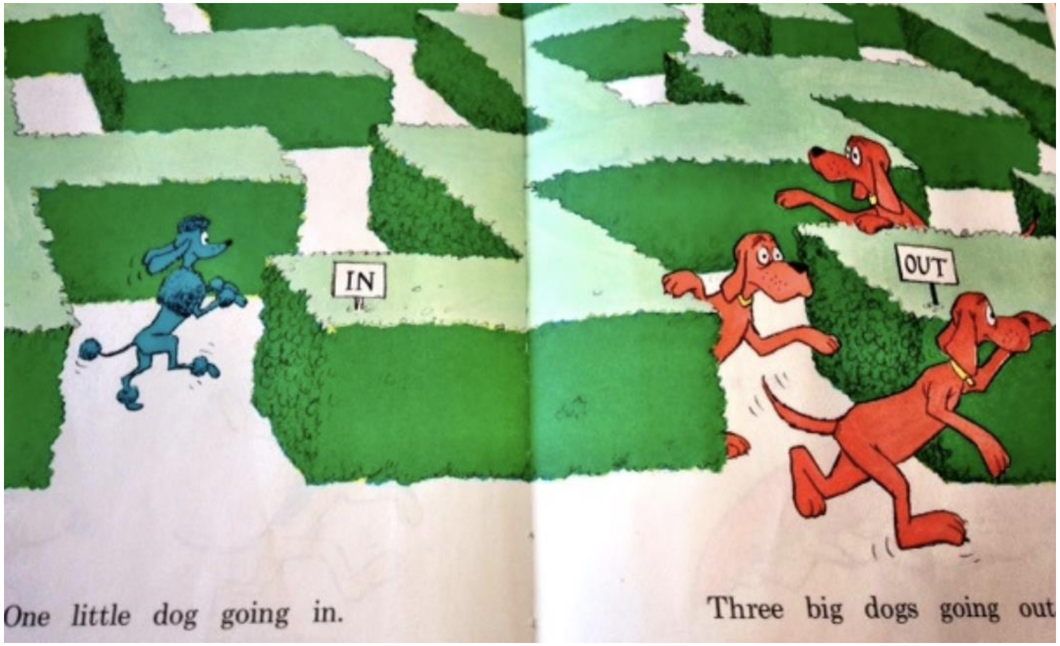
What is the S-matrix?



- All things EFT Seminar
  Feb 2, 2022

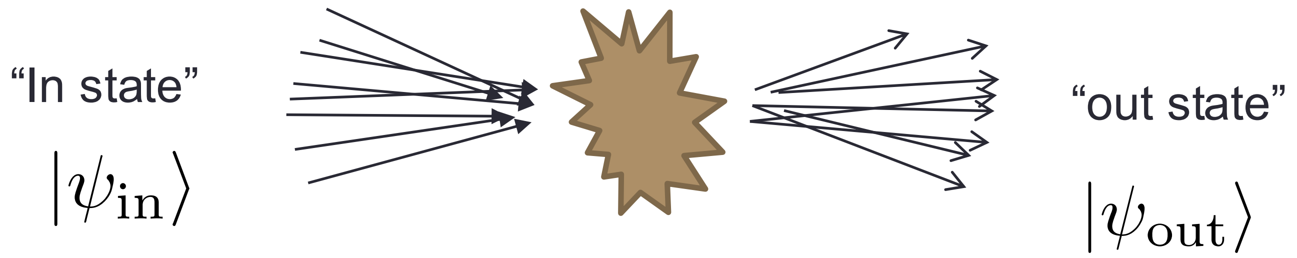
Matthew Schwartz
Harvard University

arXiv: 2109.09744 with H. Hannesdottir, A. McLeod and C. Vergu
arXiv: 2007.13747 with J. Bourjaily, H. Hannesdottir, A. McLeod and C. Vergu
arXiv:1906.03271 with H. Hannesdottir
arXiv:1810.10022 with C. Frye, H. Hannesdottir, K. Yan and N. Paul

Who cares about the S matrix?

- Hundreds of thousands of papers written about the S-matrix
- S-matrix is related to experiment (or is it?)
  - Comparing theory to nature helps us learn about nature and theory
- What are its symmetries?
  - e.g. Lorentz invariance, unitarity, dual conformal invariance
- What is its analytic structure?
  - where are its singularities?
  - Can we use them to bootstrap up all of S?
- How do we best encode its content?
  - Spinors, twistors, amplituhedron?

What is the S-matrix?



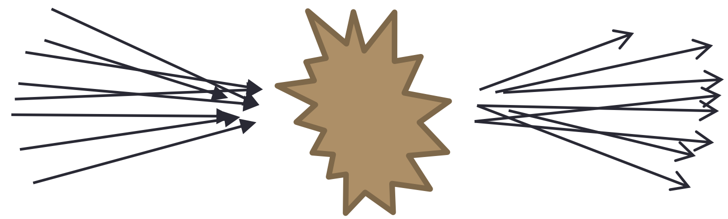
- Intuition: S should map incoming states to outgoing states
- Schrodinger picture language is most natural

$$|\psi_{\text{in}}\rangle \to e^{-iHt}|\psi_{\text{in}}\rangle$$

Attempt #1

$$S_{\text{in,out}} = \lim_{t\to\infty}\langle\psi_{\text{out}}|e^{iHt}|\psi_{\text{in}}\rangle$$

What is the S-matrix?



Attempt #1: $$S_{\text{in,out}} = \lim_{t\to\infty} \langle \psi_{\text{out}} | e^{iHt} | \psi_{\text{in}} \rangle$$

Problems:
- limit does not exist for free theory:
  - H = H_0

$$S = \lim_{t\to\infty} e^{iE_0 t} = \;???$$

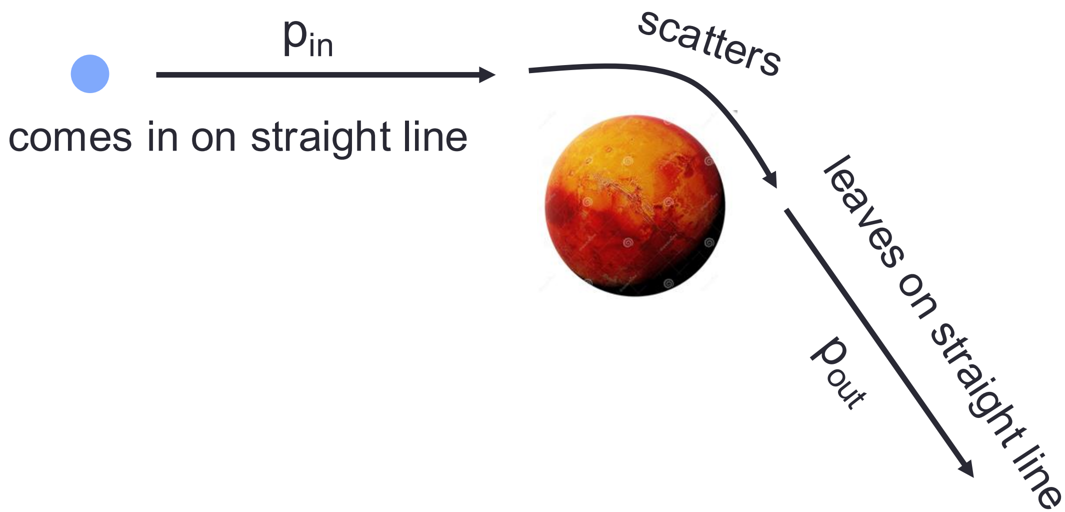
- S-matrix is trivial for energy eigenstates

$$H|\psi_{\text{in}}\rangle = E_{\text{in}}|\psi_{\text{in}}\rangle$$
$$H|\psi_{\text{out}}\rangle = E_{\text{out}}|\psi_{\text{out}}\rangle$$
$$S \sim \delta(E_{\text{in}} - E_{\text{out}})$$

(*and* limit does not exist: $S = \delta(E_{\text{in}} - E_{\text{out}}) \lim_{t\to\infty} e^{iE_{\text{out}} t} = \;???$)

What is the S-matrix?

classical scattering



- Momentum doesn't change much at very early and very late times
  - Hamiltonian is **approximately free** at early/late times
  - Momentum is a constant of motion in the free theory
  - Momentum is *not* an eigenstate of the full Hamiltonian

- S-matrix is **projection of free Hamiltonian eigenstates p_{out} onto p_{in}**

Wheeler-Heisenberg S matrix

quantum mechanics

$$\xleftarrow{\;e^{iH_0 t}\;} |\psi_{\text{in}}\rangle$$

freely evolving state (asymptotic state)

$$\xleftarrow[\;e^{iHt}\;]{} |\psi\rangle$$

state evolving with H

$$\lim_{t\to-\infty} e^{-iHt}|\psi\rangle = \lim_{t\to-\infty} e^{-iH_0 t}|\psi_{\text{in}}\rangle \qquad \lim_{t\to\infty} e^{-iHt}|\psi\rangle = \lim_{t\to\infty} e^{-iH_0 t}|\psi_{\text{out}}\rangle$$

$$|\psi\rangle = \Omega_-|\psi_{\text{in}}\rangle \qquad |\psi\rangle = \Omega_+|\psi_{\text{out}}\rangle$$

Moller operator: $\Omega_- = \lim_{t\to-\infty} e^{iHt}e^{-iH_0 t}$ $\qquad \Omega_+ = \lim_{t\to\infty} e^{iHt}e^{-iH_0 t}$

$$\Rightarrow |\psi_{\text{out}}\rangle = \Omega_+^\dagger|\psi\rangle = \underbrace{\Omega_+^\dagger\Omega_-}_{\text{S-matrix}}|\psi_{\text{in}}\rangle$$

Attempt #2:

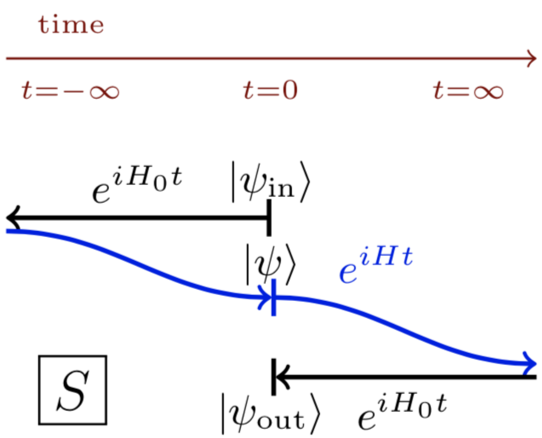
S-matrix: maps between eigenstates of free Hamiltonian

Wheeler-Heisenberg S matrix

Attempt #2: $$S = \Omega_+^\dagger \Omega_-$$

$$\Omega_\pm = \lim_{t\to\pm\infty} e^{iHt} e^{-iH_0 t}$$

Wheeler-Heinsenberg S matrix



- Now, if $H = H_0$, $S = 1$, as desired
- This is the usual **interaction picture:**
  - States evolve with H_0
  - Operators evolve with ~ "$H - H_0$"

Note that

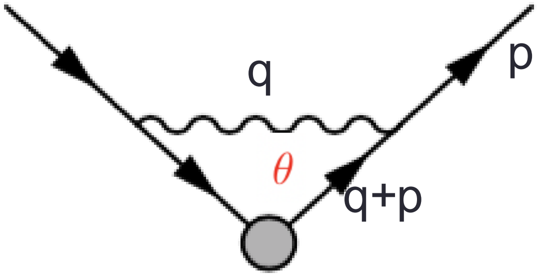
> The Wheeler-Heinsenberg S matrix is just an operator
> - not some God-given thing essential to QFT

Problems

- Haag's theorem: interaction picture doesn't exist
  - Moller operators are not unitary operator on Hilbert space (non-perturbative)
  - Moller operators don't exist order-by-order in perturbation theory due to UV divergences
- Non-perturbative S-matrix *does* exist (Haag-Ruelle, LSZ)
  - Proof uses H only (not H_0), not Moller operators
  - Uses a different definition of S
    - exact mass eigenstates created by $a_p^\dagger$ defined in terms of φ
  - Requires a mass gap, Wightman axioms
  - Cannot actually compute anything with it
- In QED or other theories with massless particles
  - S matrix is **infinite** (infrared/collinear divergences) perturbatively
  - S matrix is **zero** (exponential suppression $S \sim e^{-\frac{1}{\epsilon}} = 0$) non-pertubatively

The biggest practical problem

Infrared divergences



$$\sim \int d^4q \frac{\cdots}{(q+p)^2}$$

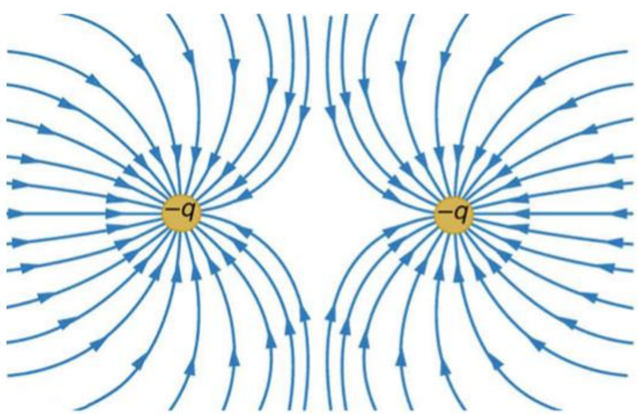
$$\sim \int \frac{d^4q}{2p\cdot q} \sim \int \frac{dE_q\, d\theta}{2E_p E_q(1-\cos\theta)} \sim \int \frac{dE_q\, d\theta}{E_q\,\theta}$$

$E_q \to 0$ soft singularity

$\theta \to 0$ collinear singularity

Physical reason for IR divergences in S

- We can never separate a charged particle from its electromagnetic field



Infrared divergences

3 approaches to making S finite

1. "Cross-section method"
    - IR divergences cancel at cross section level
    - S matrix does not need to exist

2. Redefine scattering states
    - Coherent/dressed state approach

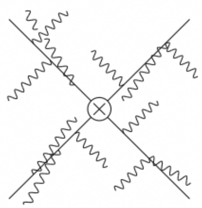
3. Redefine S
    - The Wheeler-Heinsenberg S matrix is just an operator
        - not some God-given thing essential to QFT
    - Can we find an operator with more sensible properties?
        - Yes! Using effective field theory

1. Cross-section method

When do soft divergences cancel at cross section level?

A. Bloch-Nordsieck theorem (1937)

- σ inclusive over outgoing photons ($E_q < \omega_{cut}$) is IR finite in QED
- Exploits simplicity of Abelian theory



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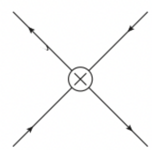


$$\times \prod_{i=1}^{\ell}\left(\sum_{j=1}^{N}\eta_j\, g\,\frac{n_j\cdot\epsilon_i}{n_j\cdot k_i}\right)\times(1+\mathcal{O}(\lambda))$$

- Fails in QCD
  - σ is still IR divergent after summing over final state gluons
  - failure related to massless charged particles
    - collinear singularities = mass singularities

1. Cross-section method

When do soft & collinear divergences cancel at cross section level?

B. KLN theorem (1963)

- Kinoshita+Sirlin (1959) $\sigma(\mu^- \to e^- \nu_\mu \bar{\nu}_e)$ divergent at m_e =0 at 1-loop

  divergence canceled by $\sigma(\mu^- \to e^- \nu_\mu \bar{\nu}_e \gamma)$
    - final state mass singularity cancellation

- Kinoshita (1962) :
  - mass singularity at m_μ=0 should cancel if we sum over initial states
- Lee and Neunberg (1963):
  - proved it
  - Sum over all possible initial and final state in an energy window

$$\int d\Pi_i d\Pi_f \theta(|E_i - E_0| - \omega_{\text{cut}}) \times \theta(|E_f - E_0| - \omega_{\text{cut}}) \times \sigma(i \to f) = \text{finite}$$

A little confusing

- Why do we need a window if energy is conserved?
- Do we really need *all possible states,* including $|i\rangle = |f\rangle$

1. Cross-section method

When do soft & collinear divergences cancel at cross section level?

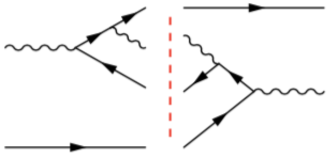
C. Frye at al (2018):

- IR divergences cancel in *any* theory when we include all initial **or** final state cuts of a given diagram

- Finite parts cancel too!
- Proof is simply unitarity
  - Probability of $|\psi_{\rm in}\rangle$ to go to anything is 1
  - Works order-by-order in pertubation theory

$$1 = 1 + 0 \cdot g + 0 \cdot g^2 + 0 \cdot g^3 + \cdots$$

- Checks are highly non-trivial
  - Involve disconnected diagrams



Bottom line: $\boxed{\text{KLN is useless}}$

1. Cross-section method

When do soft & collinear divergences cancel at cross section level?

Summary
- Cancellation of soft singularities in QED requires summing over final state photons
- Collinear singularities arise from massless charged particles
  - Cancellation requires summing over all degenerate initial and final states
  - $\sigma = 0$ to all orders
- How to handle charged initial states is an open problem

Usual procedure
1. Argue there are no massless charged particles anyway
   - misses the point
   - mass turns divergence into large log
     - how do we sum those large logs?
2. Subtract off collinear limits using splitting functions/**PDFs**
   - Observables must avoid forward scattering region
   - Assumed, not proven, to work
   - Can't compute total cross sections
   - We don't know if there's a problem if we don't know what to look for

Infrared divergences

3 approaches to making S finite

1. "Cross-section method"
   - IR divergences cancel at cross section level
   - S matrix does not need to exist

✓

2. Redefine scattering states
   - Coherent/dressed state approach

3. Redefine S
   - The Wheeler-Heinserberg S matrix is just an operator
     - not some God-given thing essential to QFT
   - Can we find an operator with more sensible properties?
     - Yes! Using effective field theory

2. Redefine scattering states

Chung (1965): replace free electrons with dressed electrons $|p^d\rangle = e^R|p\rangle$

$$R|p\rangle = e\sum_{j=1}^{2}\int \frac{d^{d-1}k}{(2\pi)^{d-1}\sqrt{2\omega_k}}\frac{p\cdot\epsilon_j(k)}{p\cdot k}a_k^{j\dagger}|p\rangle$$

soft/eikonal emission amplitudes

Then $\langle p_3^d \cdots p_n^d|S|p_1^d p_2^d\rangle$ is IR finite in QED

- Cancellation mechanism the same as in Bloch-Nordsieck
  - Amplitude minus its soft limit is soft finite

Many issues

- Moves the IR divergence problem from amplitudes to states
- Hilbert space of "coherent states" does not exist [Kibble 1968]
- No successful generalization to QCD
- No one ever explicitly computed matrix elements of dressed states

Infrared divergences

3 approaches to making S finite

1. "Cross-section method"
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   - S matrix does not need to exist

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3. Redefine S matrix

Wheeler-Heisenberg S matrix

$$S = \Omega_+^\dagger \Omega_-$$

Moller operators

$$\Omega_\pm = \lim_{t\to\pm\infty} e^{iHt} e^{-iH_0 t}$$

removes free evolution

Dollard (1970)

- IR divergence in non-realtivistic quantum mechanics is only in Coulomb phase

$$\langle \vec{p}_f | S | \vec{p}_i \rangle \sim \frac{\alpha}{(\vec{p}_i - \vec{p}_f)^2} e^{-i\alpha \frac{m}{|\vec{p}_i - \vec{p}_f|} \frac{1}{2\epsilon_{\mathrm{IR}}}}$$

- we should include Coulomb interaction in the asymptotic evolution
- replace H_0 with an asymptotic Hamiltonian

$$H_0 \longrightarrow H_{\text{asymptotic}} = H_0 + H_{\text{Coulomb}}$$

$$\Omega_\pm \longrightarrow \lim_{t\to\pm\infty} e^{iHt} e^{-iH_{\text{as}} t}$$

- What happens in the relativistic case?

3. Redefine S matrix

Wheeler-Heisenberg S matrix

$$S = \Omega_+^\dagger \Omega_-$$

Moller operators

$$\Omega_\pm = \lim_{t\to\pm\infty} e^{iHt} e^{-iH_0 t}$$

removes free evolution

Fadeev & Kulish (1970): generalization to relativistic QED

$$S_{\mathrm{FK}} = \lim_{t_\pm\to\pm\infty} e^{-R(t_+)} e^{-i\Phi(t_+)} S e^{-i\Phi(t_-)} e^{R(t_-)}$$

put R in dressed states

Φ goes in H_{as}

$$R(t) = e\sum_{j=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left[\frac{p\cdot\epsilon_j^*(k)}{p\cdot k}a_k^{j\dagger}e^{i\frac{p\cdot k}{\omega_p}t} - \frac{p\cdot\epsilon_j(k)}{p\cdot k}a_k^{j}e^{-i\frac{p\cdot k}{\omega_p}t}\right]\rho(\vec{p})$$

$$\Phi(t) = \frac{\alpha}{2}\int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3} :\rho(p)\rho(q): \frac{p\cdot q}{\sqrt{(p\cdot q)^2 - m^4}}\ln|t|$$

$$\rho(p) = \sum_s\left(a_p^{s\dagger}a_p^s - b_p^{s\dagger}b_p^s\right)$$

Showed matrix elements of S_{FK} are finite

- Same mechanism as Bloch-Nordsieck
- Same problems as Chung
  - cutoffs needed
  - no explicit calculations
  - fails for mass singularities & QCD

50 years later,
can we do better?

We want to define a **hard S-matrix** using **hard Moller opertors**

$$S_H = \Omega_+^{H\dagger} \Omega_-^H \qquad \Omega_\pm^H = \lim_{t_\pm \to \pm\infty} e^{iHt_\pm} e^{-iH_{\mathrm{as}} t_\pm}$$

How do we pick H_{as} for massless QCD?

I claim a great choice is $\boxed{H_{as} = H_{SCET}}$

Hamiltonian of Soft-Collinear Effective Theory

Has many nice properties

- Includes all non-decoupling long-distance/collinear interactions
- Hard S-matrix elements are IR finite
  - equal to Wilson coefficients
  - closely related remainder functions
- Factorization: states evolve independently of how they scatter
- Can compute hard S-matrix elements explicitly
  - Can replace explicit cutoffs with dimensional regularization
  - μ dependence drops out of observables, as usual

How do we compute S_H matrix elements?

First write

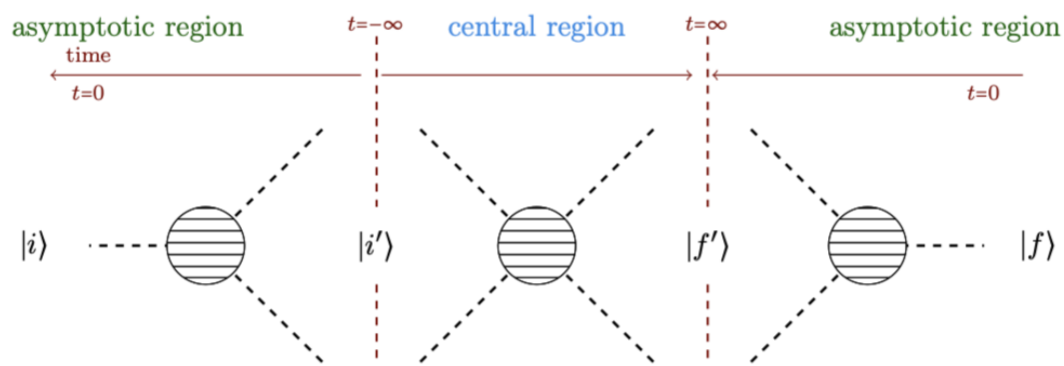
$$S_H = \Omega_+^{H\dagger}\,\Omega_-^{H} = \Omega_+^{\rm as}\,\Omega_+^{\dagger}\,\Omega_-\,\Omega_-^{\rm as\dagger} = \Omega_+^{\rm as}\,S\,\Omega_-^{\rm as\dagger}$$

Then

good-old Wheeler-Heisenberg S matrix

$$\langle\psi_{\rm out}|S_H|\psi_{\rm in}\rangle = \int d\Pi_{\psi'_{\rm out}} \int d\Pi_{\psi'_{\rm in}} \langle\psi_{\rm out}|\Omega_+^{\rm as}\,|\psi'_{\rm out}\rangle\,\underbrace{\langle\psi'_{\rm out}|\,S\,|\psi'_{\rm in}\rangle}\,\langle\psi'_{\rm in}|\,\Omega_-^{\rm as\dagger}|\psi_{\rm in}\rangle$$

usual free-theory Fock states



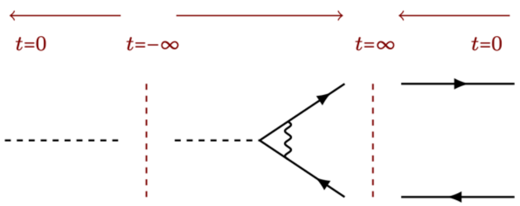
state gets "dressed" using H_{SCET}

ordinary S-matrix elements

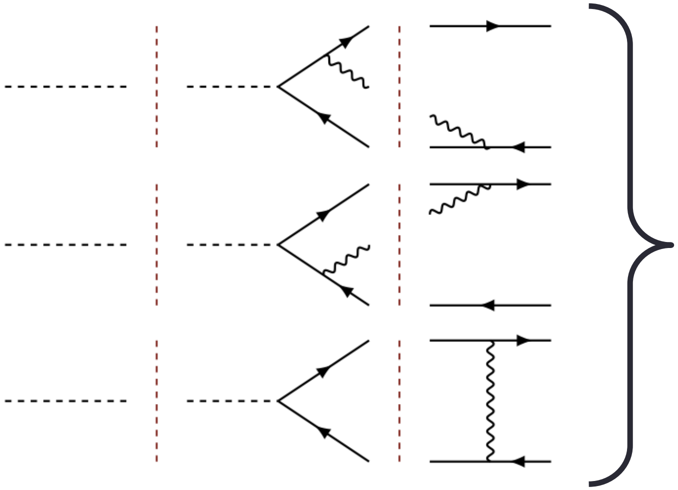
state gets "undressed" using H_{SCET}

Example: $Z \to e^+e^-$

$$L = \ln \frac{-m_Z^2}{\mu^2}$$



$$= \mathcal{M}_0 \frac{\alpha}{4\pi}\left[\frac{1}{\epsilon_{\rm UV}} - \frac{2}{\epsilon_{\rm IR}^2} - \frac{4+2L}{\epsilon_{\rm IR}} - 8 + \frac{\pi^2}{6} - L^2 + 3L\right]$$



$$= \mathcal{M}_0 \frac{\alpha}{4\pi}\left[\frac{2}{\epsilon_{\rm IR}^2} + \frac{4+2L}{\epsilon_{\rm IR}} - \frac{2}{\epsilon_{\rm UV}^2} - \frac{4+2L}{\epsilon_{\rm UV}}\right]$$

total is $$\langle e^+e^- | S_H | Z \rangle^{\overline{\rm MS}} = \mathcal{M}_0 + \mathcal{M}_0 \frac{\alpha}{4\pi}\left[-8 + \frac{\pi^2}{6} - L^2 + 3L\right]$$

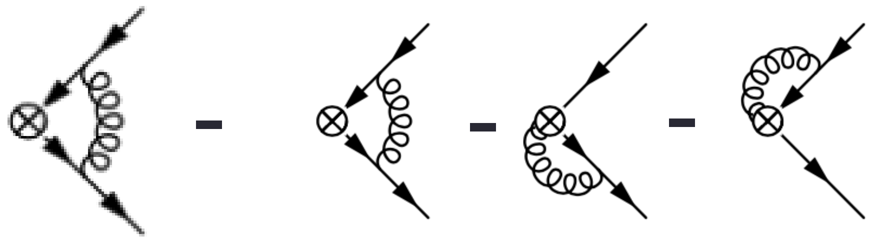
- IR finite at amplitude level!

Interpretations

$$\langle e^+e^-|\, S_H \,|Z\rangle^{\overline{\text{MS}}} = \mathcal{M}_0 + \mathcal{M}_0\frac{\alpha}{4\pi}\left[-8+\frac{\pi^2}{6}-L^2+3L\right]$$

1. Hard S matrix elements = Wilson Coefficients

- Same as Wilson coefficient in SCET



$$= \mathcal{M}_0\frac{\alpha}{4\pi}\left[-8+\frac{\pi^2}{6}-L^2+3L\right]$$

QCD soft n-collinear n̄-collinear

- Not entirely trivial
  - Graphs are different, but still scaleless
  - Could depend on scheme

New interpetation:

- Wilson coefficients usually from matching between two theories
  - Now understood as **matrix elements of a single operator**
- Justifies why we might expect Wilson coefficients to have **interesting analytic or symmetry properties**

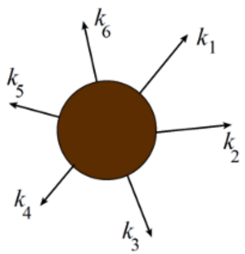
Interpretations

1. Hard S matrix elements = Remainder functions

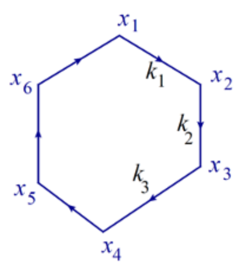
What is a remainder function?

Dual conformal invariance (DCI) is symmetry under

momenta



$k_i = x_i - x_{i+1}$



dual variables

$$x_i^\mu \to \frac{x_i^\mu}{x_i^2} \qquad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}$$

$$x_{ij} = x_i - x_j$$

DCI:
M only depends on
DCI cross ratios $u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$

- Symmetry of tree-level planar N=4
- Violated at 1-loop
  - loops in N=4 are IR divergent
  - regulating IR divergences breaks scale invariance

What's the point of a symmetry that's violated at 1-loop?

1-loop planar N=4 SYM 4-point amplitude

- Color order amplitude
- Divide out by tree-level spin structures

$$M_4^{(1)}(\epsilon) = -\frac{2}{\epsilon^2} + \frac{1}{\epsilon} M_4^{(1)}(\epsilon^{-1}) + M_4^{(1)}(\epsilon^0) + \cdots$$

where

IR divergences

$$M_4^{(1)}(\epsilon^{-1}) = -\ln\frac{\mu^2}{-s} - \ln\frac{\mu^2}{-t}$$

$$M_4^{(1)}(\epsilon^{0}) = -\ln\frac{\mu^2}{-t}\ln\frac{\mu^2}{-s} + \frac{2\pi^2}{3}$$

$$\begin{aligned} M_4^{(1)}(\epsilon^{1}) = &-\frac{\pi^2}{2}\ln\frac{-s}{u} - \frac{1}{3}\ln^3\frac{-s}{u} + \frac{\pi^2}{12}\ln\frac{\mu^2}{-s} - \frac{1}{6}\ln^3\frac{\mu^2}{-s} + \frac{\pi^2}{4}\ln\frac{\mu^2}{u} + \frac{1}{2}\ln^2\frac{-s}{u}\ln\frac{\mu^2}{u} - \frac{1}{2}\ln\frac{-s}{u}\ln\frac{-t}{u}\ln\frac{\mu^2}{u} \\ &- \ln\frac{-s}{u}\mathrm{Li}_2\frac{-s}{u} + \mathrm{Li}_3\frac{-s}{u} + \frac{7}{3}\zeta_3 + (s\leftrightarrow t) \end{aligned}$$

$$\begin{aligned} M_4^{(1)}(\epsilon^{2}) = &\frac{5\pi^2}{24}\ln^2\frac{-s}{u} + \frac{1}{8}\ln^4\frac{-s}{u} + \frac{3}{8}\ln\frac{-s}{u}\ln\frac{-t}{u} + \frac{1}{6}\ln^3\frac{-s}{u}\ln\frac{-t}{u} - \frac{1}{4}\ln^2\frac{-s}{u}\ln^2\frac{-t}{u} + \frac{\pi^2}{24}\ln^2\frac{\mu^2}{-s} - \frac{1}{24}\ln^4\frac{\mu^2}{s} - \frac{\pi^2}{2}\ln\frac{-s}{u}\ln\frac{\mu^2}{u} \\ &- \frac{1}{3}\ln^3\frac{-s}{u}\ln\frac{\mu^2}{u} + \frac{\pi^2}{8}\ln^2\frac{\mu^2}{u} + \frac{1}{4}\ln^2\frac{-s}{u}\ln^2\frac{\mu^2}{u} - \frac{1}{4}\ln\frac{-s}{u}\ln\frac{-t}{u}\ln^2\frac{\mu^2}{u} + \frac{7}{3}\zeta_3\ln^2\frac{\mu^2}{-s} + \frac{1}{2}\ln^2\frac{-s}{u}\mathrm{Li}_2\frac{-s}{u} - \ln\frac{-s}{u}\ln\frac{\mu^2}{u}\mathrm{Li}_2\frac{-s}{u} \\ &+ \ln\frac{\mu^2}{u}\mathrm{Li}_3\frac{-s}{u} - \ln\frac{-s}{u}\mathrm{Li}_3\frac{-t}{u} - \mathrm{Li}_4\frac{-s}{u} + \frac{49\pi^4}{720} + (s\leftrightarrow t) \end{aligned}$$

Looks kind of nasty...

4 point amplitude at **2-loops** has no new functions!

$$M_4^{(2)} = \frac{2}{\epsilon^4} - \frac{2}{\epsilon^3} M_4^{(1)}(\epsilon^{-1}) + \frac{1}{\epsilon^2}\left[\frac{\pi^2}{12} + \frac{1}{2} M_4^{(1)}(\epsilon^{-1})^2 - 2M_4^{(1)}(\epsilon^0)\right]$$
$$+ \frac{1}{\epsilon}\left[-\frac{\pi^2}{12} M_4^{(1)}(\epsilon^{-1}) + M_4^{(1)}(\epsilon^{-1}) M_4^{(1)}(\epsilon^0) - 2M_4^{(1)}(\epsilon^1) + \frac{\zeta_3}{2}\right] + M_4^{(2)}(\epsilon^0) + \mathcal{O}(\epsilon)$$

$$M_4^{(2)}(\epsilon^0) = \frac{1}{2}\left[M_4^{(1)}(\epsilon^0)\right]^2 - \frac{\pi^2}{6} M_4^{(1)}(\epsilon^0) - \frac{\pi^4}{120} + M_4^{(1)}(\epsilon^{-1})\left[M_4^{(1)}(\epsilon^1) - \frac{\zeta_3}{2}\right] + M_4^{(1)}(\epsilon^{-2}) M_4^{(1)}(\epsilon^2)$$

BDS Ansatz: all orders amplitude is exponential of 1-loop up to constants

$$\mathcal{M}_n^{\rm BDS} = \exp\left[\sum_L \left(\left(4\pi e^{-\gamma}\right)^\epsilon \frac{g_s^2 N_c}{8\pi^2}\right)^L \left(f^{(L)}(\epsilon) M_n^{(1)}(L\epsilon) + C^{(L)} + E_n^{(L)}(\epsilon)\right)\right]$$

$$C^{(1)} = 0 \qquad C^{(2)} = -\frac{1}{2}\zeta_2^2 \qquad E_n^{(L)}(0) = 0$$

- reduces amplitude to numbers
- not quite this simple for n=6+ legs...

BDS Ansatz

$$\mathcal{M}_n^{\text{BDS}} = \exp\left[\sum_L \left((4\pi e^{-\gamma})^\epsilon \frac{g_s^2 N_c}{8\pi^2}\right)^L \left(f^{(L)}(\epsilon) M_n^{(1)}(L\epsilon) + C^{(L)} + E_n^{(L)}(\epsilon)\right)\right]$$

- fails for n>5
- parametrize deviation from BDS with remainder functions

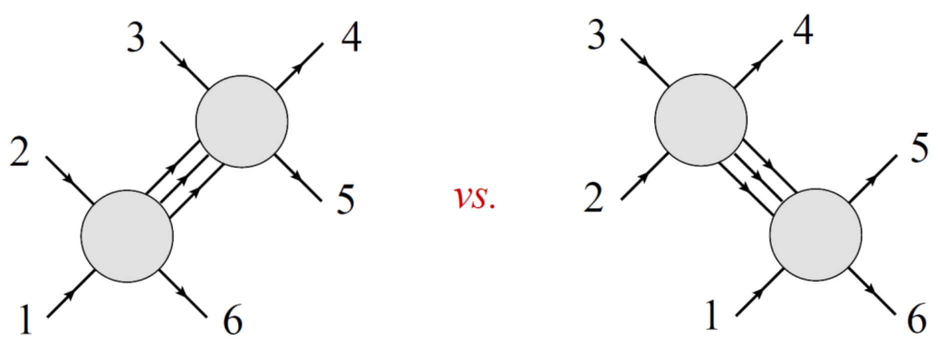
$$R_n = \ln\left[\frac{\mathcal{M}_n}{\mathcal{M}_n^{\text{BDS}}}\right]$$

- remainder functions respect dual conformal invariance
- ... but violate **Steinmann Relations**

Steinmann relations

Steinmann (1960):
amplitudes cannot have sequential discontinuities in partially overlapping channels:

$$\mathrm{Disc}_{s_{126}}\mathrm{Disc}_{s_{234}}\mathcal{I} = 0$$



- Provides important constraints
- e.g. Caron-Huot, Dixon, McLeod, von Hippel 1609.00669
  Boostrapped the 5-loop 6-particle amplitude using Steinmann

Why does it hold? When does it hold? What else holds?

BDS Ansatz

$$\mathcal{M}_n^{\text{BDS}} = \exp\left[\sum_L \left((4\pi e^{-\gamma})^\epsilon \frac{g_s^2 N_c}{8\pi^2}\right)^L \left(f^{(L)}(\epsilon) M_n^{(1)}(L\epsilon) + C^{(L)} + E_n^{(L)}(\epsilon)\right)\right]$$

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- remainder functions respect dual conformal invariance
- violate **Steinmann Relations**
  - sequential discontinuities in partially-overlapping channels must vanish

- could divide by other things, e.g. ”BDS-like”, or “minimally-normalized”:

$$\mathcal{M}_n^{\text{BDS}} \longrightarrow \mathcal{M}_n^{\text{min}} = \exp\left[\sum_L \left((4\pi e^{-\gamma})^\epsilon \frac{g_s^2 N_c}{8\pi^2}\right)^L \left(f^{(L)}(\epsilon) M_n^{(1,\text{div})}(L\epsilon) + C^{(L)}\right)\right]$$

Minimally-normalized remainder function is

$$\widehat{M}_4^{\text{BDS},(1)} = -\ln\frac{\mu^2}{-t}\ln\frac{\mu^2}{-s} + \frac{5\pi^2}{6}$$

Same as hard S-matrix element in $\overline{\text{MS}}$!

6 point 1-loop amplitude

$$\widehat{M}_6^{(1)}(\epsilon) = \sum_{\text{cycles}} \left[-\frac{1}{2}\ln^2(-s_{12}) - \ln\frac{-s_{12}}{-s_{123}}\ln\frac{-s_{23}}{-s_{123}} + \frac{1}{4}\ln^2\frac{-s_{123}}{-s_{234}} \right] \qquad (\overline{\text{MS}})$$
$$- \text{Li}_2(1-u) - \text{Li}_2(1-v) - \text{Li}_2(1-w) + 6\zeta_2 + \mathcal{O}(\epsilon)$$

conformal cross ratios

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}}, \quad v = \frac{s_{23}s_{56}}{s_{234}s_{123}}, \quad w = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$

- 1-loop used for subtraction
  - no predictions at 1-loop
- BDS-like subtraction uses

$$\widehat{M}_6^{(1)}(\epsilon) = \sum_{\text{cycles}} \left[-\ln(-s_{12})\ln(-s_{23}) + \frac{1}{2}\ln(-s_{12})\ln(-s_{45}) \right] + 6\zeta_2 \qquad \text{(BDS-like scheme)}$$

  - exponentiation involves only 2-particle invariants
    - cannot violate Steinmann relations

Different remainder schemes correspond to different subtraction schemes in SCET

- Not clear if they are all consistent
  - Counterterms must be local
- Different choices have different symmetries. Must keep exploring!

Review

- The traditional S matrix is **just an operator. It is not fundamental.**
- An S matrix is only useful if it can
  1. Predict observables
  2. Elucidate symmetries/teach us about QFT
- The traditional S-matrix subtracts off the free Hamiltonian

$$S = \Omega_+^\dagger \Omega_- \quad \Omega_\pm = \lim_{t \to \pm\infty} e^{iHt} e^{-iH_0 t}$$

  - With massless particles, this is not a great choice
    - S has singularities
    - S breaks symmetries (conformal inviariance, etc.)
- An alternative **Hard S-matrix** subtracts off all asymptotic interactions

$$S_H = \Omega_+^{H\dagger} \Omega_-^H \quad \Omega_\pm^H = \lim_{t_\pm \to \pm\infty} e^{iHt_\pm} e^{-iH_{\text{as}} t_\pm}$$

Hamiltonian of SCET

  - S-matrix elements are Wilson coefficients
  - Preserves symmetries like conformal invariance

Open questions

- What is an electron?
  - Many ways to “dress” an electron.
  - All ways give the same formal IR-divergence cancellation
  - What principle fixes the finite part of the dressing?
- Do PDFs work?
  - Singularities in forward scattering region are not reproduced by splitting functions
  - Glauber modes are needed
  - Is being inclusive over the forward region enough?
- What are the the symmetry/analyticity properties of S_H?
  - Hard S-matrix elemens are IR and UV finite
  - Still have branch points
  - Can results of S-matrix theory for massive theories be used in massless theories?
    - e.g. Steinmann relations

Steinmann relations

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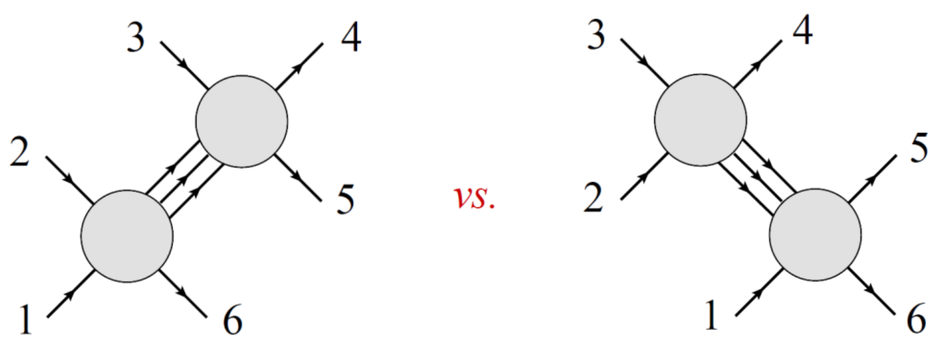
$$\mathrm{Disc}_{s_{126}}\mathrm{Disc}_{s_{234}}\mathcal{I} = 0$$



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Why does it hold? When does it hold? What else holds?

Steinmann relations

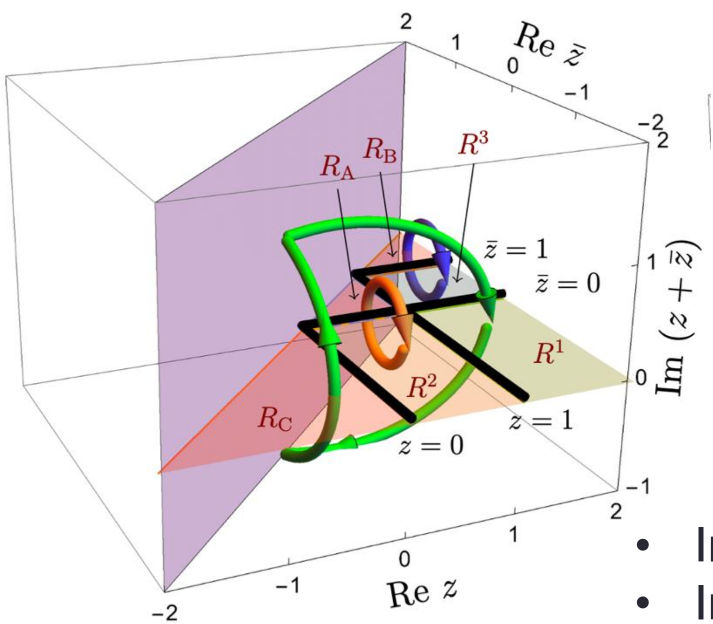
Steinmann (1960): original papers in German.

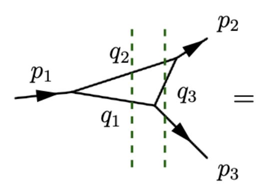
- Totally impenetrable

arXiv: 2007.13747 "Sequential Discontinuities and the Monodromy Group"

J. Bourjaily, H. Hannesdottir, A. McLeod, MDS, and C. Vergu

- Proved Steinmann using old-fashioned time-ordered pertubation theory





$$= \frac{1}{E_1 - (\omega_1+\omega_2) + i\varepsilon} \frac{1}{(E_1 - E_3) - (\omega_2+\omega_3) + i\varepsilon}.$$

- In a region where $s_{126}>0$ and $s_{234}>0$, no TOPT diagram involves both invariants

$$\mathrm{Disc}_{s_{126}}\mathrm{Disc}_{s_{234}}\mathcal{I} = 0$$

Broader lessons (for me)

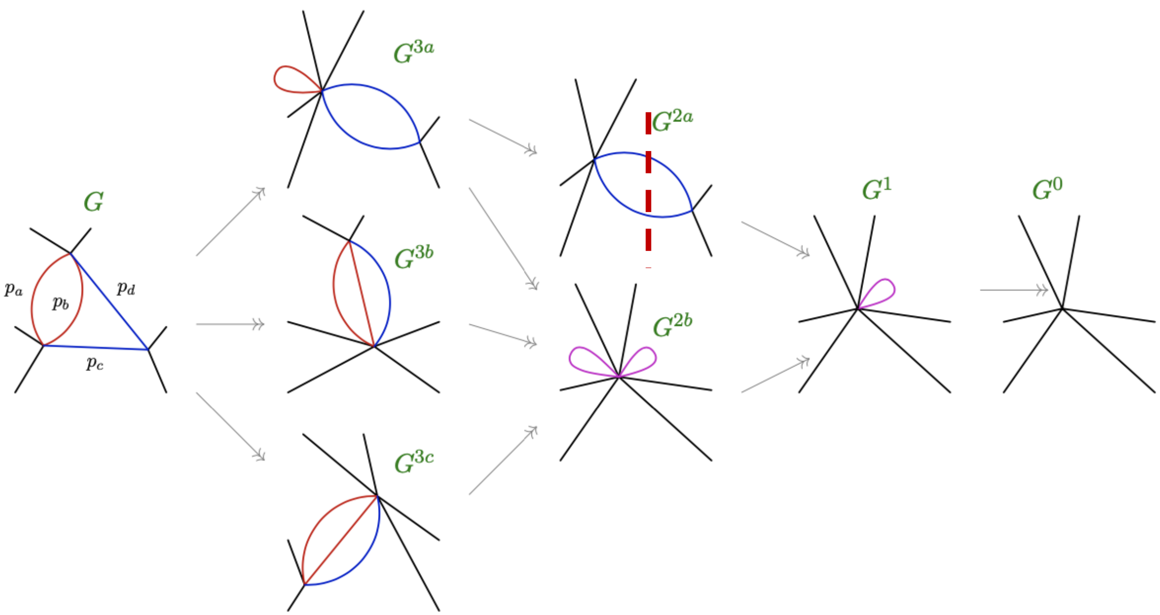
- Important to understand physical regions
- Important to understand paths of analytic continuation
- Much more to be learned about sequential discontinuties

Sequential discontinuites

- Sequential discontinuities are at the heart of the S-matrix program
  - Double, triple, etc. dispersion relations around all branch points
- Strongly constrained
- Work in progress with Hannesdottir, McLeod and Vergu.

1. Hierarchical principle (Landshoff et al 1969)

Sequential discontinuities must be in a strict hierarchy



Sequential discontinuites

- Sequential discontinuities are at the heart of the S-matrix program
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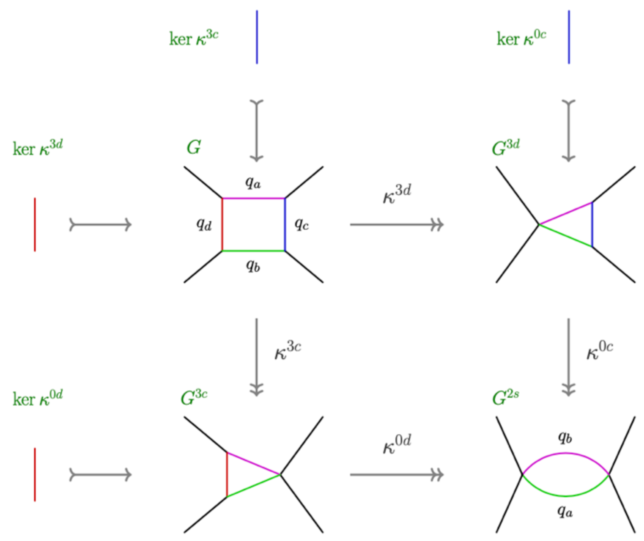
2. Pham diagrams (Pham 1967)

- Sequential discontinuities must form commutative diagrams
- Can only compute a discontinuity/monodromy using a codimension 1 contour



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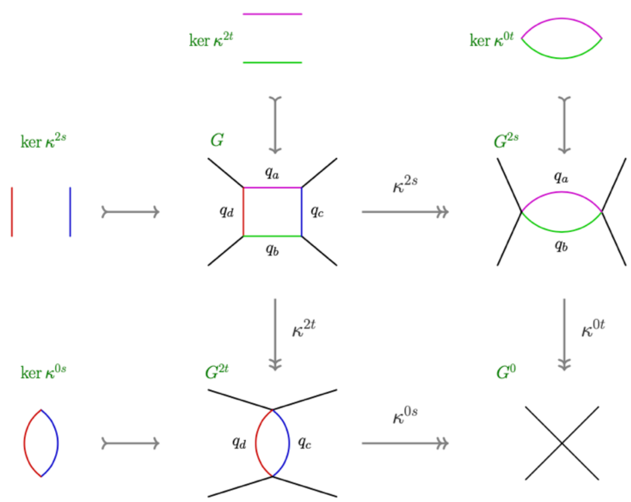

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Conclusions

Q: What is the S-matrix?
A: Just some stupid operator

Better question: What is an electron?

- After 100 years, we still don't understand what scattering is

- Infrared divergence cancellation is not even well understood
  - KLN theorem is trivial: probability of any state evolving is 1
  - Forward scattering region complicates factorization
  - Do PDFs solve the problem, or sweep it under the non-pertubative rug?

- It's possible define an S-matrix for massless particles with nice properties
  - IR and UV finite
  - Symmetries (conformal invariance)
  - Analytic properties (Steinmann relations)