Analytic Regression and the Semi-numerical Landau Boostrap

Symbology@15

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Based on

“Applications of the Landau Bootstrap” 2410.02424

“Constraints on sequential discontinuities from the geometry of on-shell spaces” 2211.07633

- By H. Hannesdottir, A. McLeod, MDS, C. Vergu

“Analytic Regression of Feynman Integrals from High-Precision Numerical Sampling”, 2507.17815

- By O. Barrera, A. Dersy, R. Husain, MDS and X. Zhang

Outline

1. Landau bootstrap

Easy ↓ Hard

Constraints	# Coeffs
All Symbols	20736
Integrability	6993
Galois symmetry	861
Physical branch cuts	161
Genealogical constraints	28
α-positive thresholds	6

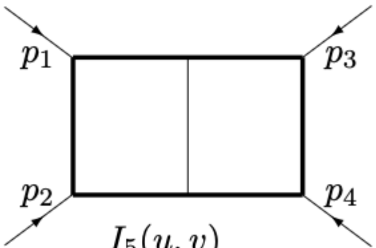
Hard ↑ Easy

2. Analytic regression with lattice reduction

Bootstrapping integrals

Q: Can we bootstrap Feynman integrals?

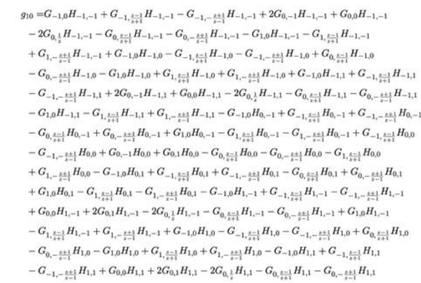
Rules: Don't integrate!



$I_5(u,v)$

$$I_5(u,v) = \int \frac{d^D k_1}{(2\pi)^D} \int \frac{d^D k_2}{(2\pi)^D} \frac{1}{[k_1^2+m^2]\,[k_2^2+m^2]\,[(k_1-p_1)^2+m^2]\,[(k_2-p_3)^2+m^2]\,[(k_1-k_2)^2]}$$

- Computed by Caron-Huot and Henn 1404.2922



1. Parametrize a finite basis

$$I_5(s,t,u,m) = \sum_{j=1}^{\text{finite}} c_j f_j(s,t,u,m)$$

- determined by singularities

2. Landau bootstrap

apply enough constraints to uniquely fix all c_j

3. Analytic regression

fit the c_j numerically

Landau Equations

$$I_G(p) = (n_{\rm int}-1)! \int_0^\infty \prod_{e\in E_{\rm int}(G)} {\rm d}\alpha_e \int \prod_{c\in \widehat{C}(G)} {\rm d}^d k_c \frac{1}{(\ell + i\varepsilon)^{n_{\rm int}}} \delta\Big(1-\sum_{e\in E_{\rm int}(G)} \alpha_e\Big)$$

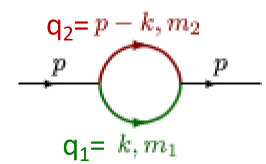
A necessary condition for a singularity is that the *integrand* is singular (ℓ=0)

$$\boxed{\ell = \sum_{e\in E_{\rm int}(G)} \alpha_e (q_e^2 - m_e^2) \;=0}$$

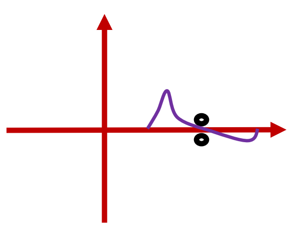
- every internal line is either on-shell (q^2=m^2) or α=0 or both

A necessary condition for a singularity of the *integral* is that poles pinch the contour

for each loop k_c: $\sum_{e\in E_{\rm int}(G^\kappa)} \alpha_e \frac{\partial}{\partial k_c}(q_e^2 - m_e^2) = 0\,.$



Double pole:



integration contour pinched between poles

- since q_e are linear in k_c

$$\Rightarrow \boxed{\sum_{e\,{\rm in\,loop}} \pm\alpha_e q_e^\mu = 0}$$

Landau loop equations

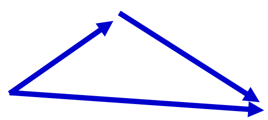
Coleman-Norton interpretation

Landau equations

$$\ell = \sum_{e\in E_{\mathrm{int}}(G)} \alpha_e(q_e^2 - m_e^2) = 0$$

$$\sum_{e\,\mathrm{in\,loop}} \pm\alpha_e q_e^\mu = 0$$

4-momenta add up to zero after rescaling by α



[Coleman and Norton 1965]

Landau diagram is interpreted as space-time diagram

- momenta are on-shell (classical)
- α_e are the proper times for propagation

More physically: singularities due to classically allowed processes

- similar to optical thoerem

Pham interpretation

Landau equations

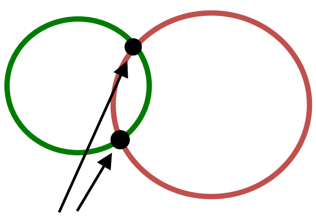
$$\ell = \sum_{e \in E_{\text{int}}(G)} \alpha_e (q_e^2 - m_e^2) = 0$$

$$\sum_{e\,\text{in}\,\text{loop}} \pm \alpha_e q_e^\mu = 0$$

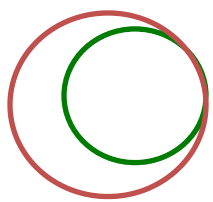
normal vectors of on-shell constraints $q^2=m^2$ are linearly dependent

on-shell constraints (Euclidean d=2)

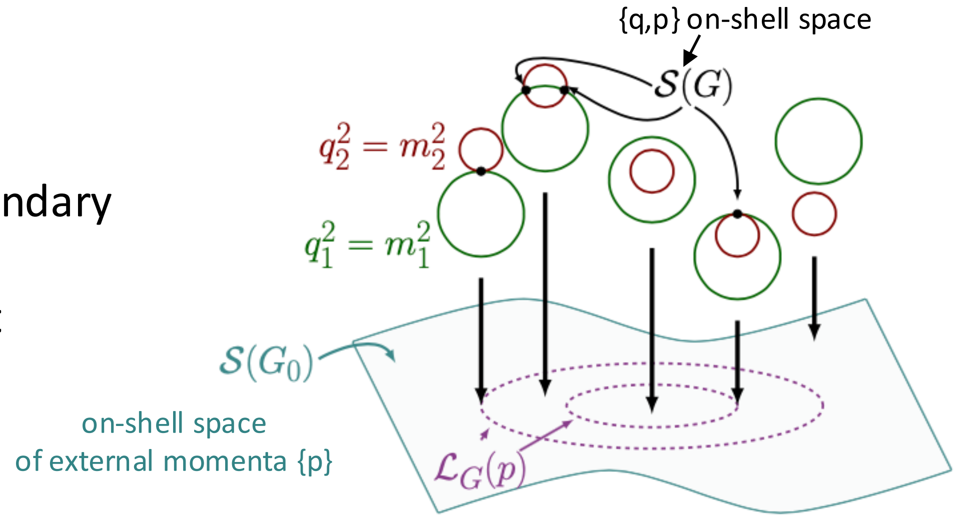
$$q_x^2 + q_y^2 = m_e^2$$



intersection satisfies both on-shell constraints



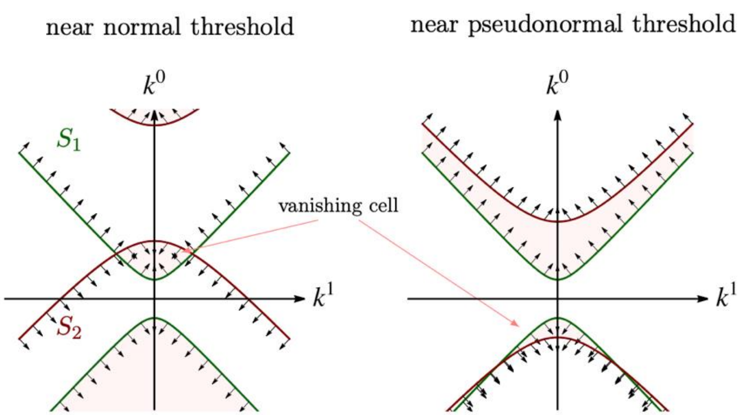
tangent on boundary of space where circles intersect



Landau variety is
the boundary of the projection map

Different kinds of singularities

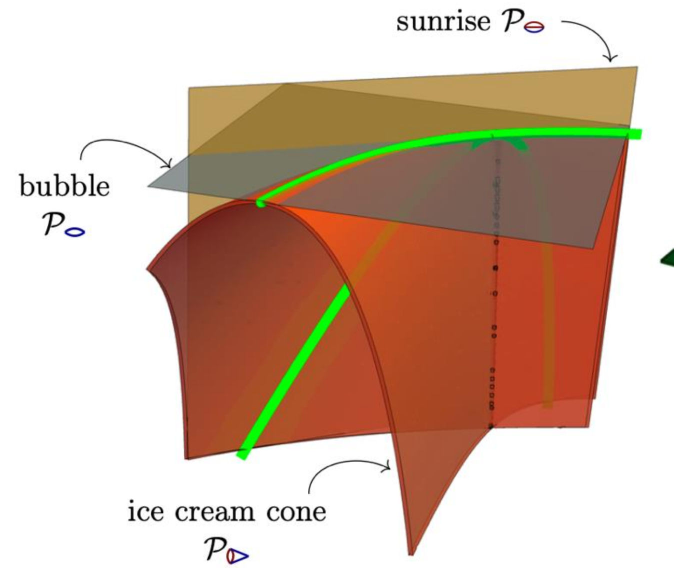
Simple pinches



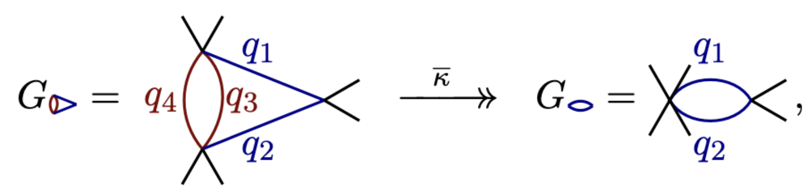
- hypersufaces meet transversely
- e.g. physical thresholds

McLeod, Hannesdottir, MDS, Vergu
arXiv:2211.07633

Non-simple pinches



- hypersufaces meet tangentially
  - e.g. sunrise in the ice-cream cone



- Permanent pinches (e.g. IR divergences)
- Pinches at infinity

Solving the Landau equations

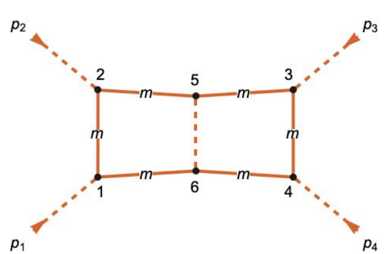
Lots of ways to solve the Landau equations

- Solve them by hand (e.g. Eden et al 1950)
- HyperInt (Panzer 2014)
- PLD (Fevola, Mizera, Telen 2013)
- BaikovLetter (Jiang et al 2024
- Recursive approach (Caron-Huot, M. Correia and M. Giroux 2024)
- Numerical implementation for any diagram (Correia, Giroux, Mizera 2024) SOFIA

Input

```
diag =
 {{{{1, 2}, m1}, {{2, 5}, m2}, {{3, 5}, m3}, {{3, 4}, m4}, {{4, 6}, m5},
     {{1, 6}, m6}, {{5, 6}, m7}}, {{1, M1}, {2, M2}, {3, M3}, {4, M4}}} //.
   m7 :> 0 //. m_ :> m //. M_ :> 0
FeynmanPlot[diag]

{{{{1, 2}, m}, {{2, 5}, m}, {{3, 5}, m}, {{3, 4}, m}, {{4, 6}, m},
  {{1, 6}, m}, {{5, 6}, 0}}, {{1, 0}, {2, 0}, {3, 0}, {4, 0}}}
```



Output

```
candidateSingularities =
  SOFIA[diag, SolverBound → Infinity];
% // TableForm

Total runtime= 2.86441

mm
s12
s23
s12 + s23
mm - s12
mm - s23
4 mm - s12
4 mm - s23
4 mm s12 + 4 mm s23 - s12 s23
mm^2 s12 - 2 mm s12 s23 - 4 mm s23^2 + s12 s23^2
mm s12^2 + 4 mm s12 s23 - s12^2 s23 + 4 mm s23^2 - s12 s23^2
```

Finds all singularities (simple & non-simple)

Finite basis

Alphabet = {letters}

- A letter L can appear in the symbol
- Integral can **only** have singularities when L=0

$$\mathcal{S}[\mathcal{I}] = L_{i_1} \otimes \ldots \otimes L_{i_n}$$

Landau's original paper determined if singularities were logarithmic or square-root

- Not so easy to use, since singularities may appear multiple times on multiple sheets
- Still true that singuarites are always either logarithmic or square root

singularities

{s,t,s-t}

alphabet

$$\left\{ s, t, s-t, \frac{\sqrt{s-t} - \sqrt{s}}{\sqrt{s-t} + \sqrt{s}} \right\}$$

$$\left\{ s=0, \frac{1}{s}=0 \right\}$$

logarithmic

s

$$\left\{ s=0, \frac{1}{t}=0, \quad s=t, t, \frac{1}{s} \right\}$$

logarithmic square root

- Galois symmetry
  - invariance under i ->-i

  or $\sqrt{\cdot} \to -\sqrt{\cdot}$
- $\sqrt{s-t}$ not a valid letter

Finite basis

Location of Landau singularities

$$s = 4m^2, \quad s \to \infty,$$
$$t = 4m^2, \quad t \to \infty,$$
$$m^2 = 0,$$
$$s = 0, \qquad t = 0, \qquad m^2 \to \infty\,,$$
$$s + t = 0, \qquad st - 4m^2 s - 4m^2 t = 0\,.$$

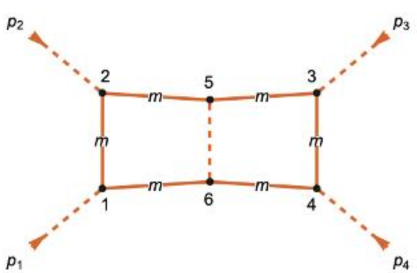
symbol alphabet

$$\tilde{A} = \Big\{u, v, 1+u, 1+v, u+v, 1+u+v, \frac{\beta_u - 1}{\beta_u + 1}, \frac{\beta_v - 1}{\beta_v + 1}, \frac{\beta_{uv} - 1}{\beta_{uv} + 1}, \frac{\beta_{uv} - \beta_u}{\beta_{uv} + \beta_u}, \frac{\beta_{uv} - \beta_v}{\beta_{uv} + \beta_v}, \frac{\beta_{uv} - \beta_u \beta_v}{\beta_{uv} + \beta_u \beta_v}\Big\}$$

$$\beta_u = \sqrt{1+u},\ \beta_v = \sqrt{1+v}\ ;\beta_{uv} = \sqrt{1+u+v}.$$

- SOFIA can also produce the alphabet (may be larger than needed)

- Length of symbol is <= 2 x loop order



2 loop outer-mass double box

- 12 letters
- 12^4 = 20,736 symbol entries

$$\mathcal{S}[\mathcal{I}] = \sum c_{\{i\}} L_{i_1} \otimes L_{i_2} \otimes L_{i_3} \otimes L_{i_4}$$

Finite basis!

Landau Boostrap

$$\mathcal{S}[\mathcal{I}] = \sum c_{\{i\}} L_{i_1} \otimes L_{i_2} \otimes L_{i_3} \otimes L_{i_4}$$

How can we fix the coefficients?

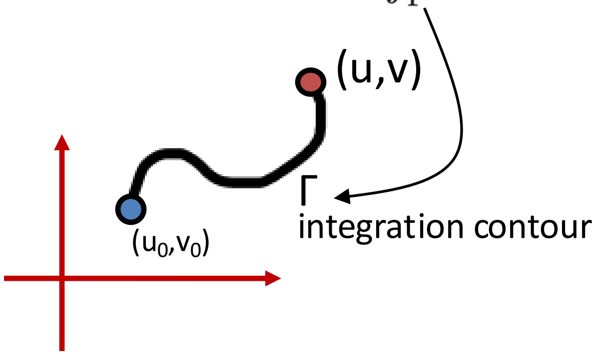
- Integrability
- Galois symmetry
- α-positivity
- First-entry conditions
- Last-entry conditions
- Geneological constraints
  - Sequential discontinuities
  - Cluster adjacency conditions
- Regions analysis
  - Soft, collinear, Regge limits
  - Often don't work diagram-by-diagram
- Direct calculation

Sequences of four letters	20736
Integrable weight-four symbols	6993
Galois symmetry	861
Physical logarithmic branch cuts	161
Genealogical constraints	28
Only algebraic α-positive thresholds	6
Threshold expansion in t	1

Integrability

- Every interated dlog integral has a symbol

$$f(u,v;\Gamma) = c_{i_1,i_2,\dots,i_n} \int_\Gamma d\ln L_{i_1} \circ \cdots \circ d\ln L_{i_n}\,. \longrightarrow \sum c_{i_1,i_2,\dots,i_n}\, L_{i_1} \otimes \cdots \otimes L_{i_n}$$



- Not every symbol corresponds to a function

- For f(u,v,Γ) to be a function, must be independent of local path deformations

$$\Rightarrow \quad [\partial_u, \partial_v] f = 0$$

- Derivatives only act on the last entry of the symbol (end of integration contour)

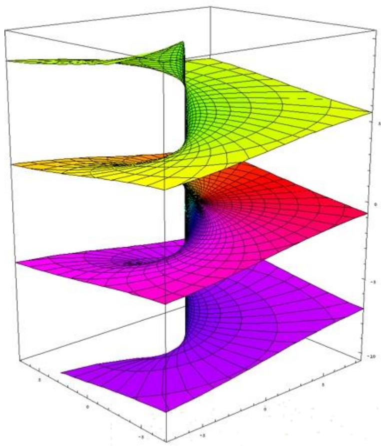
$$\partial_u[S \otimes K \otimes L] = (\partial_u \ln L)[S \otimes K]$$

$$\partial_v\partial_u[S \otimes K \otimes L] = (\partial_v\partial_u \ln L)[S \otimes K] + (\partial_u \ln L)(\partial_v \ln K) S$$

$$[\partial_u, \partial_v][S \otimes K \otimes L] = \underbrace{[(\partial_u \ln L)(\partial_v \ln K) - (\partial_v \ln K)(\partial_u \ln L)]}_{\text{must vanish}} S$$

must vanish
(integrability condition)

α positivity



- Symbol encodes all branch points, even on unphysical sheets

$$I_G(p) = (n_{\text{int}}-1)! \int_0^\infty \prod_{e\in E_{\text{int}}(G)} \mathrm{d}\alpha_e \int \prod_{c\in\widehat{C}(G)} \mathrm{d}^d k_c \frac{1}{(\ell+i\varepsilon)^{n_{\text{int}}}} \delta\Big(1-\sum_{e\in E_{\text{int}}(G)} \alpha_e\Big)$$

actual Feynman integral on physical sheet is over positive α

- Discontinuities/monodromies act on first entry of the symbol

$$I = \int \omega_1 \int \omega_2 \ldots \int \omega_n$$
$$dI = \omega_1 \int \omega_2 \ldots \int \omega_n$$

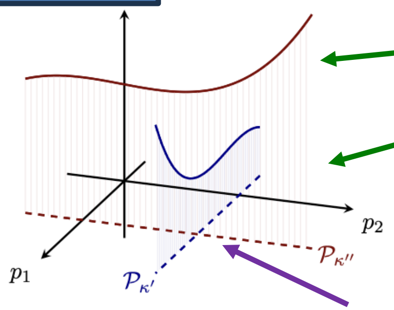
- Singularity for physical momenta (physical sheet) ⟷ singularity of first entry
  - See if α > 0 in solutions to Landau equations: constrain first symbol entries

Genealogical constraints

Which symbol entries can be adjacent?

- Adjacent means sequential discontinuities are possible

Pham approach



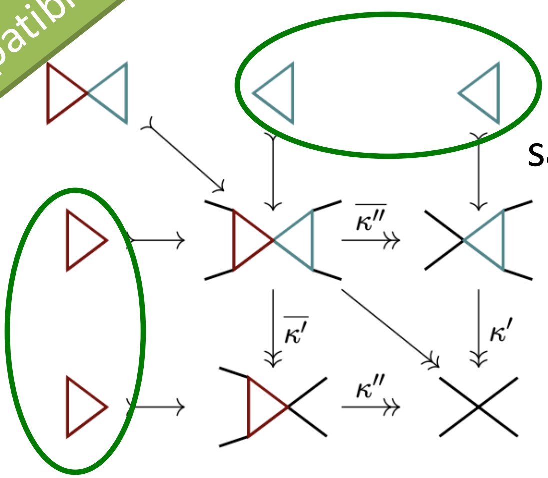
on-shell surfaces may not intersect in *internal momenta*

- vanishing cell from first monodromy doesn't intersect integration contour of second
- sequential monodromy vanishes

$$\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa'}}\right)\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa''}}\right)I_G(p)=0$$

Singular surfaces intersect transversaly in external momenta

compatible

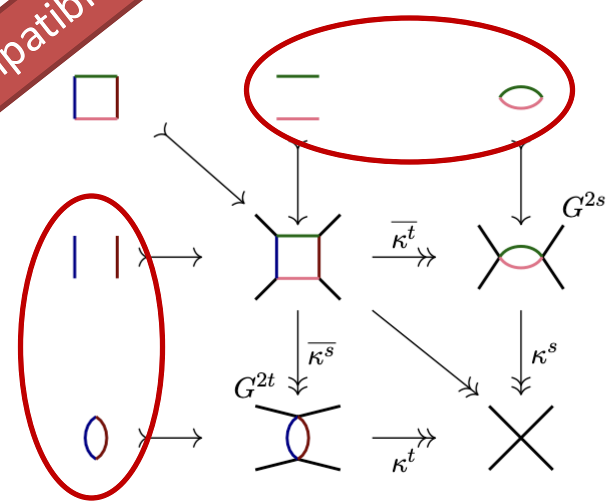


same

same

incompatible

different



different

- e.g. Steinmann relations

Outer-mass double box

Landau boostrap

Constraints	# Coeffs
All Symbols	20736
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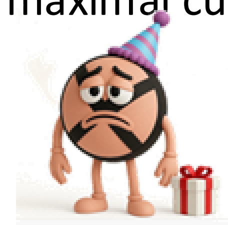
α-positive thresholds	6

- Impose as many constraints as you can
- Work from easy generic stuff (integrability) to hard
- Also need rational prefactor
  - SOFIA can compute prefactor from maximal cut
  - Not part of the symbol
- Final steps may involve computing integral in a limit

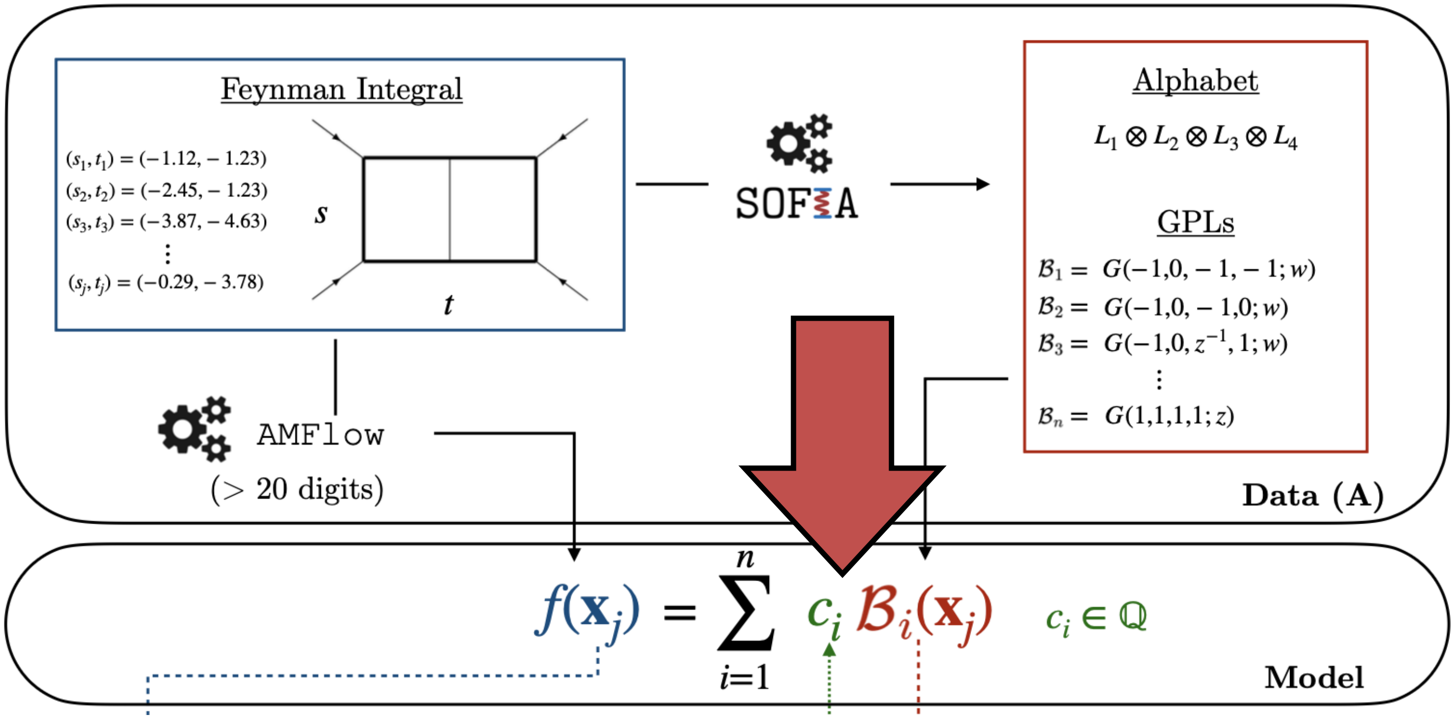


- Landau bootstrap works!

$$\mathcal{S}(\tilde{\mathcal{I}}_{\text{dbox}}) = -L_6 \otimes \frac{L_1}{L_3} \otimes L_6 \otimes L_9 - L_6 \otimes \frac{L_1}{L_3} \otimes L_9 \otimes L_6 + L_6 \otimes L_6 \otimes \frac{L_1 L_2}{L_3 L_5} \otimes L_9 + L_6 \otimes L_9 \otimes \frac{L_2}{L_5} \otimes L_6 + L_6 \otimes L_6 \otimes L_8 \otimes L_6 + L_6 \otimes L_9 \otimes L_8 \otimes L_9 + L_7 \otimes L_{10} \otimes \frac{L_2}{L_5} \otimes L_6 + L_7 \otimes L_{10} \otimes L_8 \otimes L_9 + L_7 \otimes L_7 \otimes \frac{L_1}{L_5} \otimes L_9 + L_7 \otimes L_7 \otimes L_8 \otimes L_6 \,.$$

- Agrees with Caron-Huot & Henn ‘14

Numerical bootstrap



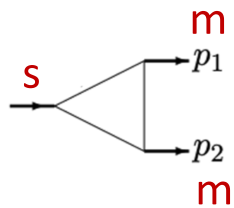
Start with finite basis

can we just fit the c_j numerically?

Requirements

1. Evaluate f(x) numerically to high precision
2. Evaluate B(x) to high precision
3. Solve the equation without losing precision

1-loop triangle

exact answer: $T =$ 

$= \frac{x}{m^2(1-x^2)}\left(-\frac{\pi^2}{3} + 4G(0,-1,x) - 2G(0,0;x)\right)$

one invariant $x = \frac{1-\sqrt{1+\frac{4m^2}{s}}}{1+\sqrt{1+\frac{4m^2}{s}}}$

Alphabet (from Sofia)

3 letters

$$A = \{L\} = \{m, s, 4m^2 - s\} \longleftrightarrow \{x, 1+x, 1-x\}$$

(Sofia computes)

Total runtime= 0.042703

$$-\frac{1}{16\,\pi^4\,\sqrt{(4\,m^2 - s)\,s}}$$

rational prefactor

Basis is

$$T(x) = \sum_{i=1}^{17} c_i \mathcal{B}_i(x) = P(x) \times [c_{ij} L_i \otimes L_j + (a_j + b_j \pi) L_j + d_1 \pi^2 + d_2 \zeta_3]$$

9 terms weight 2

6 terms weight 1

2 numbers

17 constants

- Need T(x) and $B_i(x)$ to high precision

Numerical Feynman Integration

- AMFlow: numerical evaluation for Feynman loop integrals to high precision

Integration-by-Parts (IBP) [Chetyrkin, Tkachov, 1981]

Rational functions — Master integrals (basis)

$$\mathcal{I}(p_i, m_e) = \int \prod_{j=1}^{L} \mathrm{d}^D k_j \frac{N(p_i, k_j)}{\prod_{e=1}^{E}(q_e^2 - m_e^2 + i\varepsilon)} \Rightarrow \mathcal{I} = \sum_a R_a(p_i \cdot p_j, m_e)\, \mathrm{MI}_a$$

Auxiliary mass flow method (AMFLow) [Liu, Ma, 1801.10523; 2107.01864; 2201.11669; 2201.11637]

1. Introduce an auxiliary mass η to some of the propagator denominators
2. Set up closed differential equations w.r.t η using IBPs
3. Solve the differential equations numerically with boundary conditions $\eta \to \infty$

Timing on 1-loop triangle

- each phase space point takes 5-10 CPU-min for 30 signficiant digits

Numerical basis functions

$$T(x) = \sum_{i=1}^{17} c_i \mathcal{B}_i(x) = P(x) \times [c_{ij} L_i \otimes L_j + (a_j + b_j \pi) L_j + d_1 \pi^2 + d_2 \zeta_3]$$

- Rational prefactor evaluation is instant
- Weight-2 symbols can be integrated into closed form expressions
    - FiberSymbol in Polylogtools can do this
    - Numerical evaluation is very fast

Integrals of highter weight symbols are not always known

- Can always integrate numerically along a path

$$\tilde{B} = \sum_{i=1}^{|\tilde{A}|} c_{i_1,i_2,i_3,i_4} \int_0^1 d\log L_{i_4}(\lambda_4) \int_0^{\lambda_4} d\log L_{i_3}(\lambda_3) \int_0^{\lambda_3} d\log L_{i_2}(\lambda_2) \int_0^{\lambda_2} d\log L_{i_1}(\lambda_1)\,,$$

- Individual terms may be path dependent, but final result is integrable
- Can do the first and last integral ananlytically
    - Reduce weight by 2: speeds up integration tremendously
- Sometimes analytic integrals can be so complicated that it is faster to do the integral numerically
    - More work needed on efficient numerical evaluation

Matrix inversion

$$T(x) = \sum_{i=1}^{17} c_i \mathcal{B}_i(x)$$

- Evaluate both sides of this equation at 17 points
- Solve 17 linear equations for coefficients
  - i.e. invert the matrix

$$M_{ij} = \mathcal{B}_i(\mathbf{x}_j) \Rightarrow c_i = (M^{-1})_{\mathrm{ij}} \cdot T(x_j)$$

Problems:

1. No way to impose that c_j are rational
   - Cannot every resolve functions that differ by irrational constants

$$2\,[\ln x] + 3\,[\pi \ln x] = (1+\pi)[\ln x] + \left(2 + \frac{1}{\pi}\right)[\pi \ln x]$$

   - Even with infinite precision, will not find the exact analytic answer

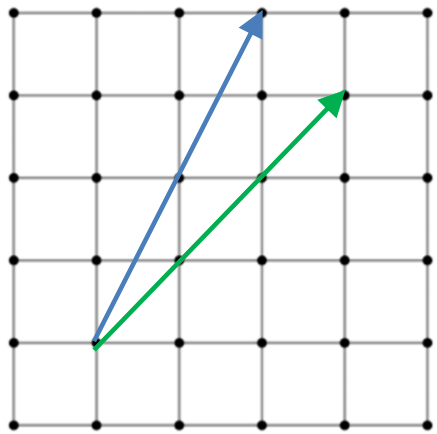
2. Matrix inversion loses precision very fast
   - Controlled by condition number

$$\kappa(M) = \|M\| \|M^{-1}\|\,.$$

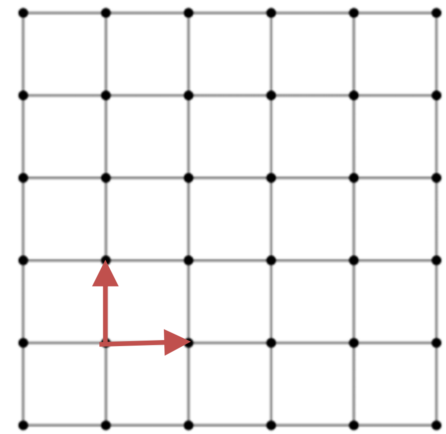
   - Generally our matrices will be ill-conditioned
     (they come from smooth functions)

Lattice reduction

vectors span a lattice



$$\vec{u}_1^2 + \vec{u}_2^2 = 24$$



$$\vec{v}_1^2 + \vec{v}_2^2 = 2$$

- Lattice reduction finds another set of vectors for same lattice
- Can minimize some norm (length of lattice vectors)
- NP-Hard problem: no polynomial-time algorithm for truly best solution
- Efficient algorithms exist to find what is almost always the minimum

Lattice reduction

Rational number coefficients can be fit for numbers using lattice reduction

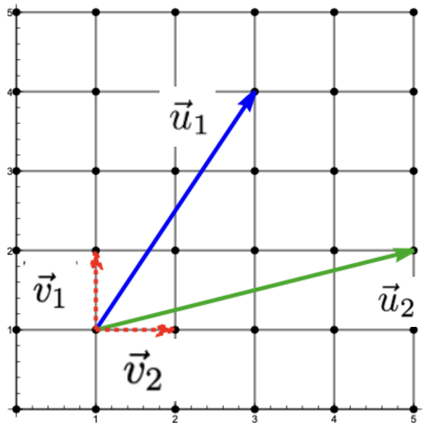
$$f = -\int_0^1 du \int_0^1 dv \frac{\log(1-uv) + v\log(1-u)}{uv} = \frac{\pi^2}{6} + \zeta_3 \,. = c_1\pi^2 + c_2\zeta_3 = \vec{c}\cdot\vec{v}$$

f=2.847 ⟷ Assume 4 digits known ⟷ π^2=9.870 ζ_3=1.202

Multiply by 10^3 and put into a matrix

$$\begin{pmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vec{u}_3 \end{pmatrix} = \begin{pmatrix} 10^3 f & 1 & & \\ 10^3\pi^2 & & 1 & \\ 10^3\zeta_3 & & & 1 \end{pmatrix} = \begin{pmatrix} 2847 & 1 & 0 & 0 \\ 9870 & 0 & 1 & 0 \\ 1202 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[\text{lattice reduction}]{} \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{pmatrix} = \begin{pmatrix} 0 & -6 & 1 & 6 \\ 3 & -11 & 5 & -15 \\ 62 & 4 & -2 & 7 \end{pmatrix}$$

vectors span a lattice



- Lattices are the same so v in the span of u

$$\vec{v}_1 = -6\vec{u}_1 + \vec{u}_2 + 6\vec{u}_3.$$

First component

$$0 = 10^3 \times (-6f + \pi^2 + 6\zeta^3) \quad ✓$$

Precision requirements

Rational number coefficients can be fit for numbers lattice reduction

$$f = -\int_0^1 du \int_0^1 dv \frac{\log(1-uv) + v\log(1-u)}{uv} = \frac{\pi^2}{6} + \zeta_3 = c_1\pi^2 + c_2\zeta_3 = \vec{c}\cdot\vec{v}$$

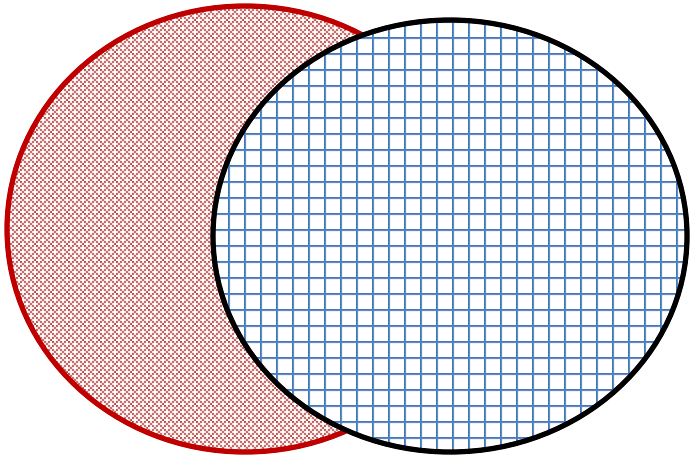
$$\vec{v} = (\pi^2, \zeta_3)$$

Q: how many digits of f are required to fit rational c_1 and c_2?

assume size of c's

$$c_1 \sim c_2 \lesssim 10^R$$

- Multiple solutions imples that $\vec{c}\cdot\vec{v} = 0$
- Multiply by GCD so c_1 and c_2 are integers
  - There are $(10^R)^n = 10^{2R}$ vectors (c_1,c_2)
- Assume d digits of precision on v
  - $\vec{c}\cdot\vec{v}$ produce 10^{2R} d digit numbers
- There are only 10^d d-digit numbers in all
- Need precision $d > nR$ to fit pure numbers
  - Information content must be sufficient



Precision requirements

For fitting *functions* we can sample at multiple points

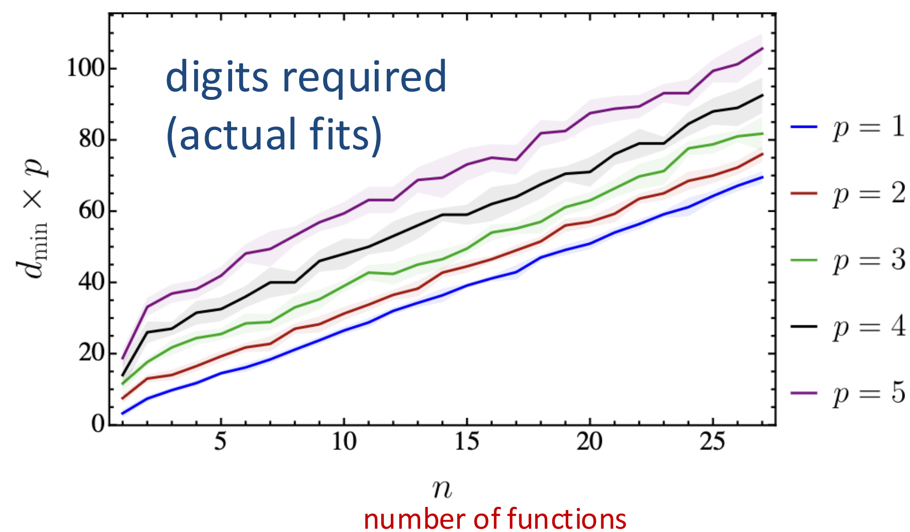
$$f(x) = G(0,1;x) - G(1,-1;x) \qquad x_1 = 4/10, x_2 = 9/10$$

$$\mathcal{B}(x) = \{G(1,0;x), G(0,1;x), G(0,-1;x), G(1,-1;x)\}$$

$$M = \text{round}\; 10^s \left(\begin{array}{c|c} f(\mathbf{x_1}) \cdots f(\mathbf{x_p}) & \\ \mathcal{B}_1(\mathbf{x_1}) \cdots \mathcal{B}_1(\mathbf{x_p}) & 10^{-s}\, \mathbb{I}_{n+1} \\ \vdots & \\ \mathcal{B}_n(\mathbf{x_1}) \cdots \mathcal{B}_n(\mathbf{x_p}) & \end{array} \right) = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \\ \mathbf{v}_5 \end{pmatrix} = \begin{pmatrix} -35 & -24 & 1 & 0 & 0 & 0 & 0 \\ 92 & 154 & 0 & 1 & 0 & 0 & 0 \\ -45 & -129 & 0 & 0 & 1 & 0 & 0 \\ 36 & 75 & 0 & 0 & 0 & 1 & 0 \\ -10 & -106 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- With p points and d digits
  - net digits of information is p × d
- Expected digits needed

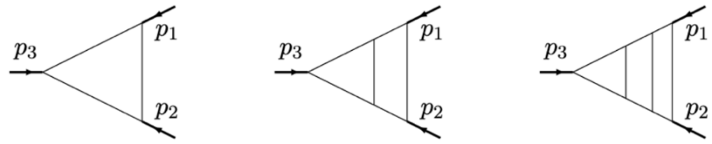
digits of precision req'd → $d \lesssim \frac{nR}{p}$ ← # basis functions (n), size of integers (R), number of points (p)



We can trade off digits of precision for number of points
- Will work even if only a few digits of precision are available!

Example 1: Triangles

Triangle ladder diagrams



$$z\bar{z} = p_2^2/p_1^2,$$
$$(1-z)(1-\bar{z}) = p_3^2/p_1^2$$

exact results known

$$T_1(z) = \frac{1}{z-\bar{z}}\left[2\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \log(z\bar{z})\log\left(\frac{1-z}{1-\bar{z}}\right)\right],$$

$$T_2(z) = \frac{1}{(1-z)(1-\bar{z})(z-\bar{z})}\Big[6\mathrm{Li}_4(z) - 6\mathrm{Li}_4(\bar{z}) - 3\log(z\bar{z})\left(\mathrm{Li}_3(z) - \mathrm{Li}_3(\bar{z})\right)$$
$$+\frac{1}{2}\log^2(z\bar{z})\left(\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})\right)\Big],$$

$$T_3(z) = \frac{1}{(1-z)^2(1-\bar{z})^2(z-\bar{z})}\Big[20\mathrm{Li}_6(z) - 20\mathrm{Li}_6(\bar{z}) - 10\log(z\bar{z})\left(\mathrm{Li}_5(z) - \mathrm{Li}_5(\bar{z})\right)$$
$$+\log^2(z\bar{z})\left(\mathrm{Li}_4(z) - \mathrm{Li}_4(\bar{z})\right) - \frac{1}{6}\log^3(z\bar{z})\left(\mathrm{Li}_3(z) - \mathrm{Li}_3(\bar{z})\right)\Big]$$

Full alphabet (from SOFIA)

$$A_{1,2} = \left\{z\bar{z},\ (1-z)(1-\bar{z}),\ z-\bar{z},\ \frac{\bar{z}}{z},\ \frac{1-z}{1-\bar{z}}\right\}$$

Simplified alphabet (quicker for testing)

$$A_3^\star = \left\{z\bar{z},\ (1-z)(1-\bar{z}),\ \frac{\bar{z}}{z},\ \frac{1-z}{1-\bar{z}}\right\}$$

Allow all possibilities up to weight 6

Weight-0: 1

Weight-1: $G(a_1,x),\ \pi$

Weight-2: $G(a_1,a_2,x),\ \pi\times G(a_1,x),\ \zeta_2$

Weight-3: $G(a_1,a_2,a_3,x),\ \pi\times G(a_1,a_2,x),\ \zeta_2\times G(a_1,x),\ \zeta_3$

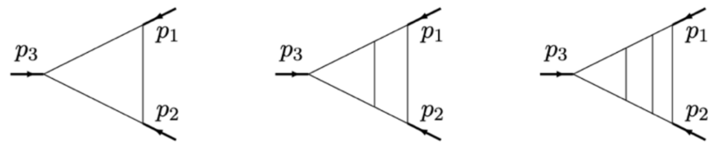
Weight-4: $G(a_1,a_2,a_3,a_4,x),\ \pi\times G(a_1,a_2,a_3,x),\ \zeta_2\times G(a_1,a_2,x),\ \pi^3\times G(a_1,x),$
$\zeta_3\times G(a_1,x),\ \zeta_4$

Weight-5: $G(a_1,a_2,a_3,a_4,a_5,x),\ \pi\times G(a_1,a_2,a_3,a_4,x),\ \zeta_2\times G(a_1,a_2,a_3,x),$
$\pi^3\times G(a_1,a_2,x),\ \zeta_3\times G(a_1,a_2,x),\ \zeta_4\times G(a_1,x),\ \zeta_5,\ \zeta_2\times\zeta_3$

Weight-6: $G(a_1,a_2,a_3,a_4,a_5,a_6,x),\ \pi\times G(a_1,a_2,a_3,a_4,a_5,x),\ \zeta_2\times G(a_1,a_2,a_3,a_4,x),$
$\pi^3\times G(a_1,a_2,a_3,x),\ \zeta_3\times G(a_1,a_2,a_3,x),\ \zeta_4\times G(a_1,a_2,x),\ \zeta_5\times G(a_1,x),$
$\zeta_2\zeta_3\times G(a_1,x),\ \pi^5\times G(a_1,x),\ \zeta_6,\ \zeta_3^2$

Example 1: Triangles

Triangle ladder diagrams



Pick random points in (unphysical) Euclidean region $0 < z < \bar{z} < 1$

- Makes basis functions real
- Imaginary numbers are fine, just technically complicated python implemntation

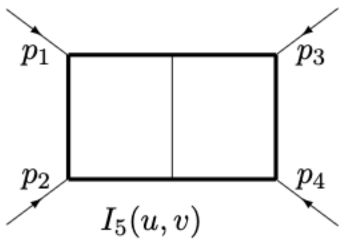
Diagram	AMFLOW point time	Transcendental weights	# points sampled	Basis size	Reduction time	
One-loop	15.6 CPU-min	≤ 2	5	full(32) simplified(26) uniform(25)	<1s <1s <1s	3 digits
Two-loop	1.1 CPU-h	≤ 4	100 100 60	full(488) simplified(393) uniform(366)	9.6 min 10.7 min 3.5 min	20 digits
Three-loop	5.7 CPU-h	≤ 6	- - 200	full(1373) simplified(972) uniform(806)	- - 1.1 h	20 digits

Results with timing

- Rate limiting step is AMFlow (computation of full function)
- Could be sped up (additonal points much faster than first)

Example 2: Double box

Loop



12 independent letters

$$\tilde{A} = \left\{u, v, 1+u, 1+v, u+v, 1+u+v, \frac{\beta_u - 1}{\beta_u + 1}, \frac{\beta_v - 1}{\beta_v + 1}, \frac{\beta_{uv} - 1}{\beta_{uv} + 1}, \frac{\beta_{uv} - \beta_u}{\beta_{uv} + \beta_u}, \frac{\beta_{uv} - \beta_v}{\beta_{uv} + \beta_v}, \frac{\beta_{uv} - \beta_u\beta_v}{\beta_{uv} + \beta_u\beta_v}\right\}$$

with $\beta_u = \sqrt{1+u}$, $\beta_v = \sqrt{1+v}$ and $\beta_{uv} = \sqrt{1+u+v}$.

- 12^4 = 20,736 weight-4 symbols +(?) lower weight terms
- square root letters are hard to integrate analytically

method 1:
rationalize the square roots

$$u = \frac{(1-w^2)(1-z^2)}{(w-z)^2} \quad \text{and} \quad v = \frac{4wz}{(w-z)^2},$$

$$\tilde{A}_2 = \{w, z, 1 \pm w, 1 \pm z, w \pm z, 1 \pm wz, 1 \pm w \mp z + wz\}\,.$$

- Now symbols can be integrated analytically
  - Takes FiberSymbol hours to integrate
  - Result is hundreds or thousands of terms
- GiNaC can get numbers out, but very slow

method 2:
numerically integate along a contour

- integrate first and last symbol analytically
- need to be careful with branch cuts
  - euclidean region requires some thought
- using integrable contributions helps a lot

Example 2: Double box

Landau boostrap

Analytic regression

relatively easy

Easy ↓ Hard

Constraints	# Coeffs
All Symbols	20736
Integrability	6993
Galois symmetry	861
Physical branch cuts	161
Genealogical constraints	28
α-positive thresholds	6

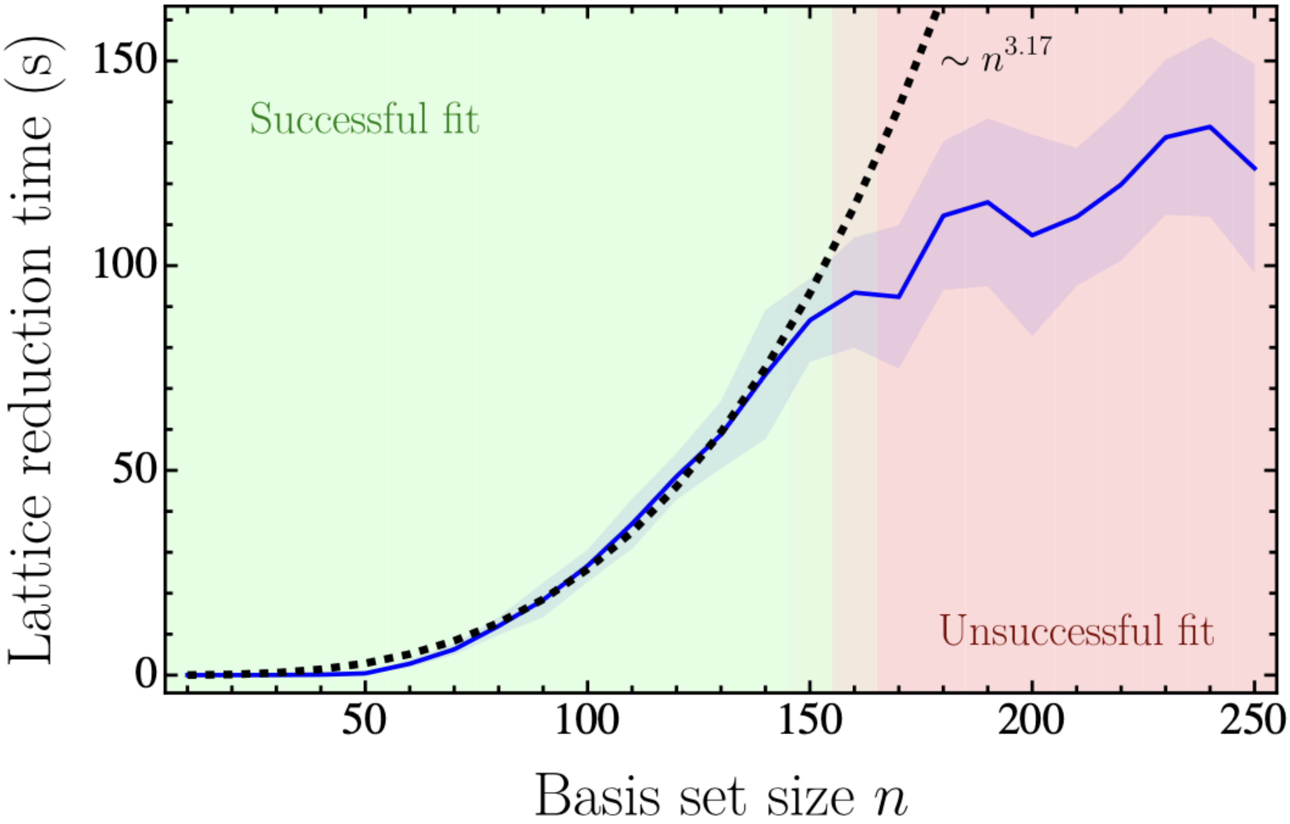
Hard ↑ Easy

✓ 200pts 17 digits, $t \backsim 30$s

✓ 200pts, 8 digits, $t < 1$s

✓ 200pts, 6 digits, $t < 1$s

Double box: limitations



Should work with more functions

- With our compute, hard to succeed with more than n $\lesssim$ 200

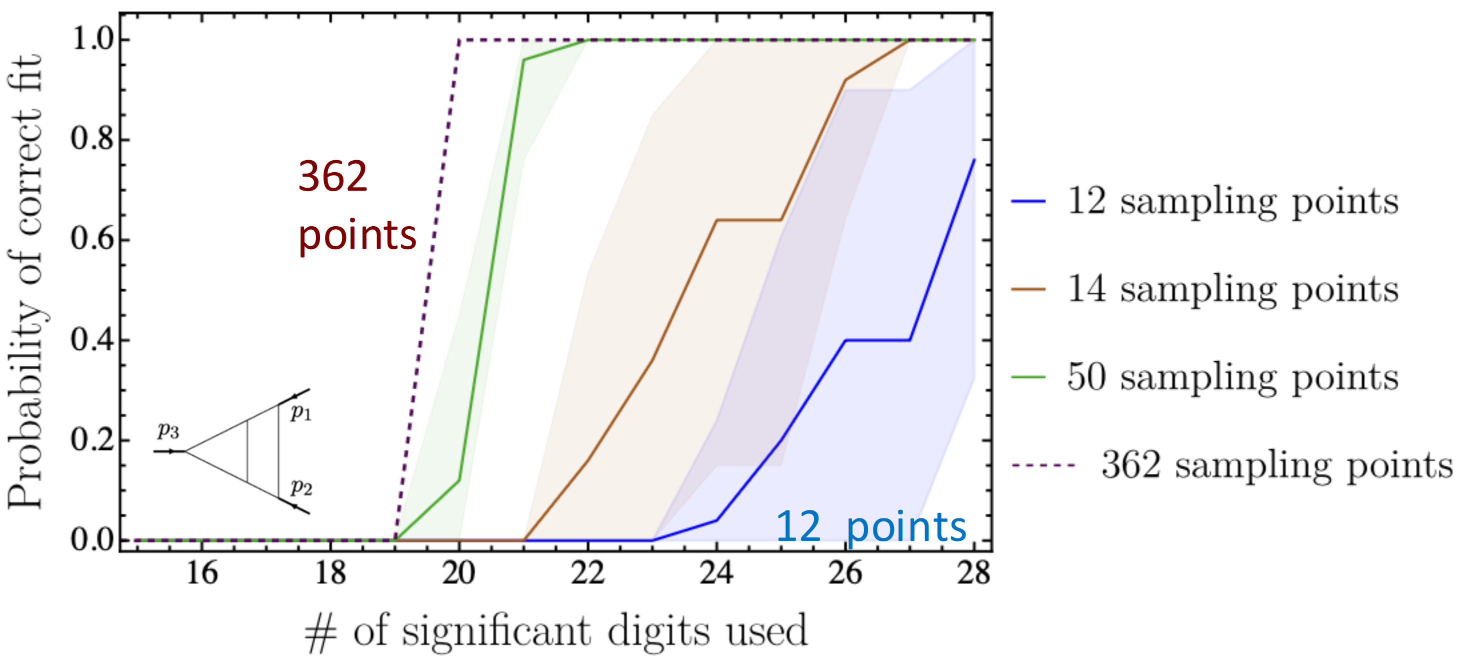
Choosing points

- Before we said you can get away with fewer digits if you use more points

basis functions

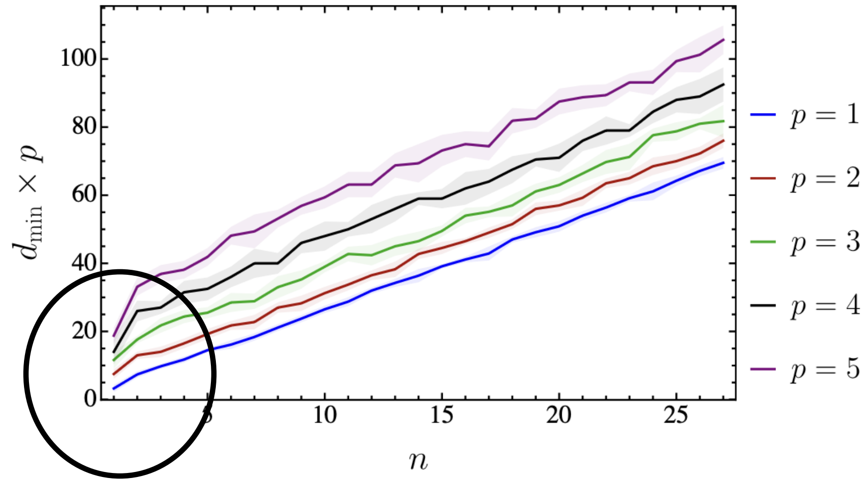
size of integers

digits of precision req'd → $$d \lesssim \frac{nR}{p}$$ ← number of points



- Can never succeed below some digit lower bound
- Why did scaling fail?

Choosing points



offset near d=0

$$d_{\min} \approx R_{\text{eff}} \frac{n}{p} + d_0$$

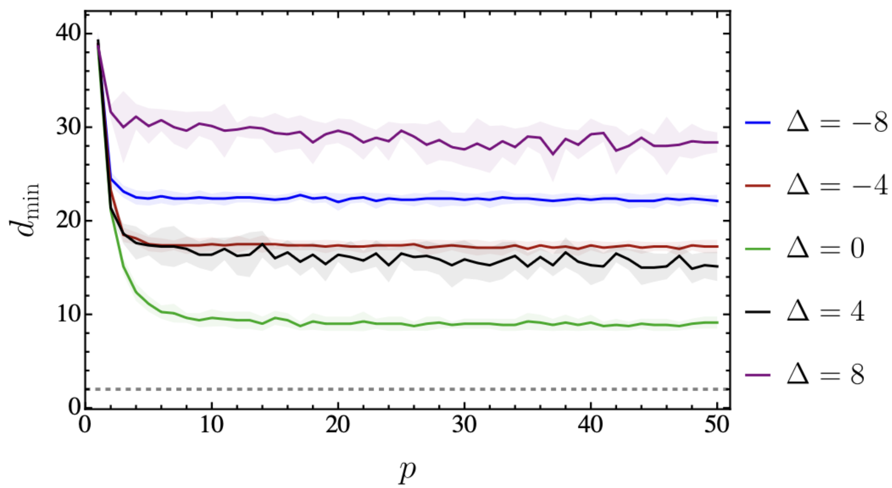
- will never work with 1 or 2 digits

$$f(p_j) = \{103.2, \quad 2.5, \quad 2.3\}$$

unrecovarable information loss if we truncate to 2 digits

choose points in a range

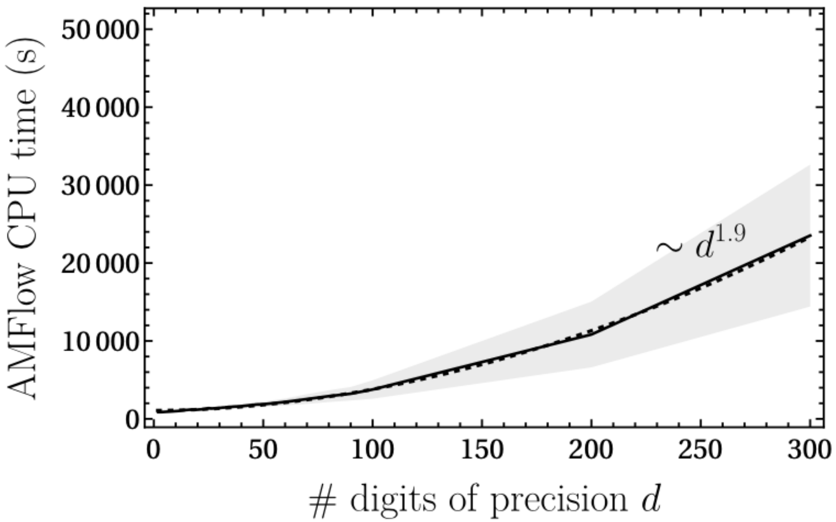
$$\tfrac{1}{2} - 10^{\Delta} \leq x \leq \tfrac{1}{2} + 10^{\Delta}$$



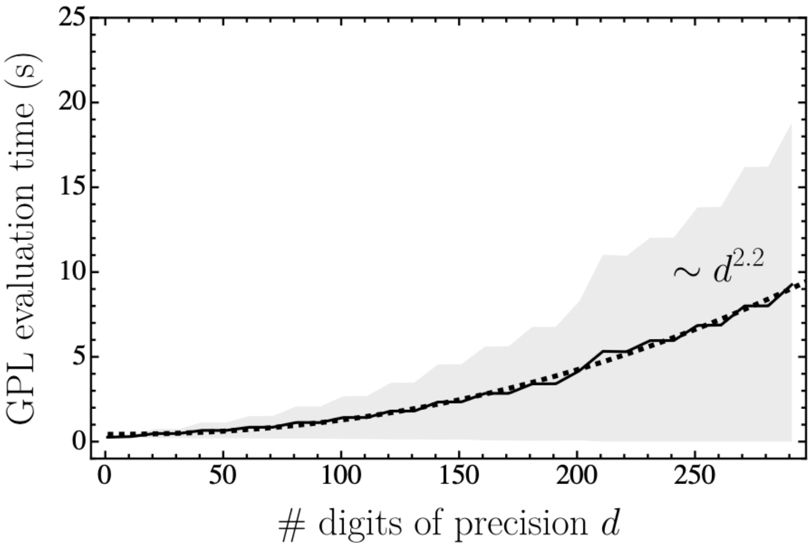
- want to choose points
  - not too close (lose information)
  - not too far (need more digits)
  - Need 10-20 digits at least

Double box: timing

AMFlow + Kira (for IBP)



GPL evaluation (GINSH)



$$t(p,n)/\mathrm{ns} \approx \underbrace{10^9 \cdot p \cdot d_{\min}^2(n)}_{\textsc{AMFlow}} + \underbrace{10^4 \cdot n \cdot p \cdot d_{\min}^2(n)}_{\textsc{Ginsh}} + \underbrace{p \cdot n^4}_{\text{fitting}},$$

- Trading digits for points makes scaling go from quadradic to linear!

Application to EECs

Toward the Analytic Bootstrap of Energy Correlators
Jianyu Gong[a,b], Andrzej Pokraka[c], Kai Yan[a,b], Xiaoyuan Zhang[d]
2509.22782

Consider some types of energy-energy correlators (unequal energy weights)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x_{12}\cdots \mathrm{d}x_{(N-1)N}} \equiv \sum_m \sum_{1\le i_1,\cdots i_N\le m} \int \mathrm{d}\sigma_m \times \prod_{1\le k\le N} \frac{E_{i_k}}{Q} \prod_{1\le j<l\le N} \delta\left(x_{jl} - \frac{1-\cos\theta_{i_j i_l}}{2}\right).$$

Uses of lattice reducion

- Find linear dependence among basis functions

$$G(z) = D(z) + a_{0,11}\, g_1(z) + a_{0,12}\, g_2(z) + a_{1,12}\, g_3(z) + a_{2,12}\, g_4(z) + c_{1,12}\, g_5(z) + d_{2,12}\, g_6(z)$$

$$\begin{aligned} g_3(z) = \frac{1}{(z-1)^2 z^2 (\bar z-1)^2 \bar z^2}\Big[& z^4\bar z^2 - z^4\bar z + 8z^3\bar z^3 - 14z^3\bar z^2 + 8z^3\bar z - z^3 + z^2\bar z^4 - 14z^2\bar z^3 \\ &+ 24z^2\bar z^2 - 14z^2\bar z + z^2 + \big(2z^4\bar z^3 - 3z^4\bar z^2 + z^4\bar z + 2z^3\bar z^4 - 8z^3\bar z^3 + 9z^3\bar z^2 - 4z^3\bar z + z^3 \\ &- 3z^2\bar z^4 + 9z^2\bar z^3 - 12z^2\bar z^2 + 9z^2\bar z - 3z^2 + z\bar z^4 - 4z\bar z^3 + 9z\bar z^2 - 8z\bar z + 2z + \bar z^3 - 3\bar z^2 \\ &+ 2\bar z\big)\log((z-1)(\bar z-1)) + \big(-2z^4\bar z^3 + 3z^4\bar z^2 - z^4\bar z - 2z^3\bar z^4 + 8z^3\bar z^3 - 9z^3\bar z^2 + 2z^3\bar z \\ &+ 3z^2\bar z^4 - 9z^2\bar z^3 + 6z^2\bar z^2 - z\bar z^4 + 2z\bar z^3\big)\log(z\bar z) - z\bar z^4 + 8z\bar z^3 - 14z\bar z^2 + 8z\bar z - \bar z^3 + \bar z^2\Big] \end{aligned}$$

- Treat $\{g_i\}$ as transcendental basis and $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}$ as coefficients to determine
- (1). Run lattice reduction among $\{g_i(z)\}$ to get a linear-independent basis $\{\tilde g_i(z)\}$
- (2). Run lattice reduction among both $G(z)$ and $\{\tilde g_i(z)\}$

In this case, 2 numerical points and 13 digits fix all six parameters

Pros and cons: Landau bootstrap

Landau boostrap

- Can eliminate large swaths of symbols with physical constraints
- Don’t need to do integrals
- Leads to new deep understanding in what amplitdues are

- Requires subtle understanding of singularities
  - Analytic structure of amplitudes
  - Branch points, euclidean regions
  - Algebraic geometry
- Rational prefactors not fixed by singularities alone
- Often still requires some integration at the end

Pros and cons: analytic regression

Analytic regression with lattice reduction

- Easy to automate
- Can work for any functions
  - not just polylogs with symbols
  - elliptic polylarithms? no problem!
  - cross sections, EECs, etc.
- Can find linear dependencies
- Can trade digits of accuracy for points
  - Scaling $t \sim n^2$ or worse to $t \sim n$

- Becomes computationally challenging above $n \sim 200$
  - AMFlow scales like $(\text{\# digits})^2$
  - lattice reduction scales like $(\text{\# constants})^4$
- Minimum number of digits needed ($\sim$ 5 or 6)

$$f(\mathbf{x}_j) = \sum_{i=1}^{n} c_i \, \mathcal{B}_i(\mathbf{x}_j) \qquad c_i \in \mathbb{Q}$$

→ problem solved

Conclusions

Happy Birthday Symbol