Renormalons, Instantons, and the Failure of Perturbation Theory

DESY

Hamburg, Germany

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Based on

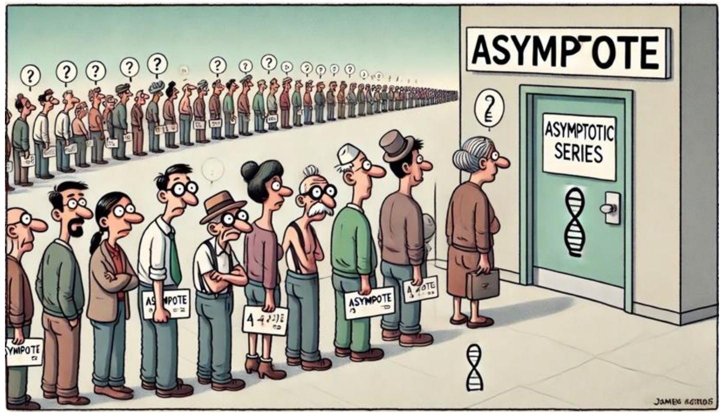
arXiv:2512.09042 "Asymptotic Behavior of Diagram Classes",
Luen Clingerman, **MDS**

arXiv:2410.07351 "Renormalons as Saddle Points",
Arindam Bhattachary, Jordan Cotler, Aurelian Dersy, **MDS**

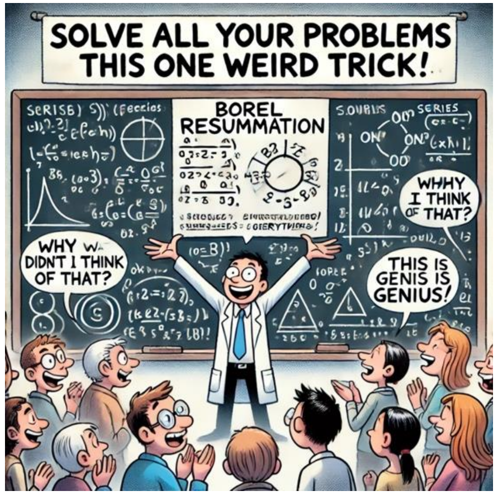
arXiv:2402.18633 "The Collective Coordinate Fix",
Arindam Bhattachary, Jordan Cotler, Aurelian Dersy, **MDS**

Outline

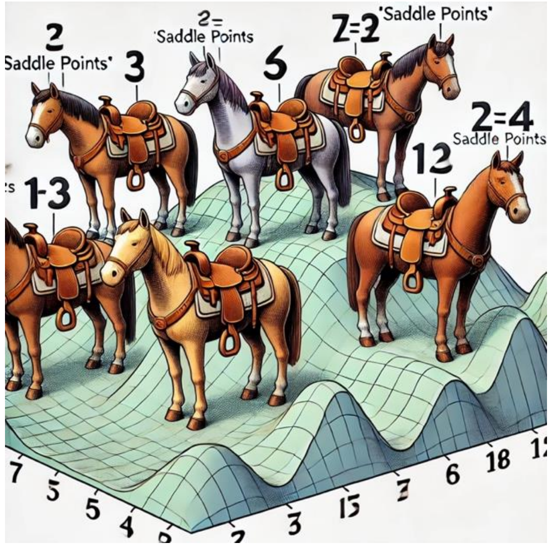
1. Asymptotic series



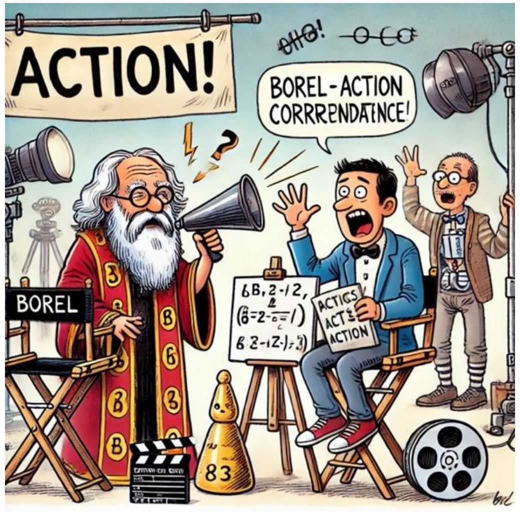
2. Borel resummation



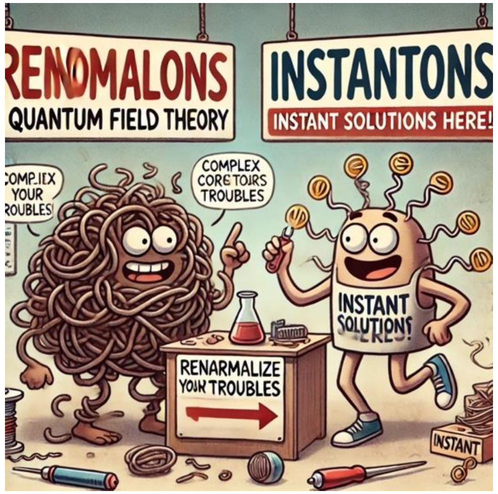
3. Saddle points



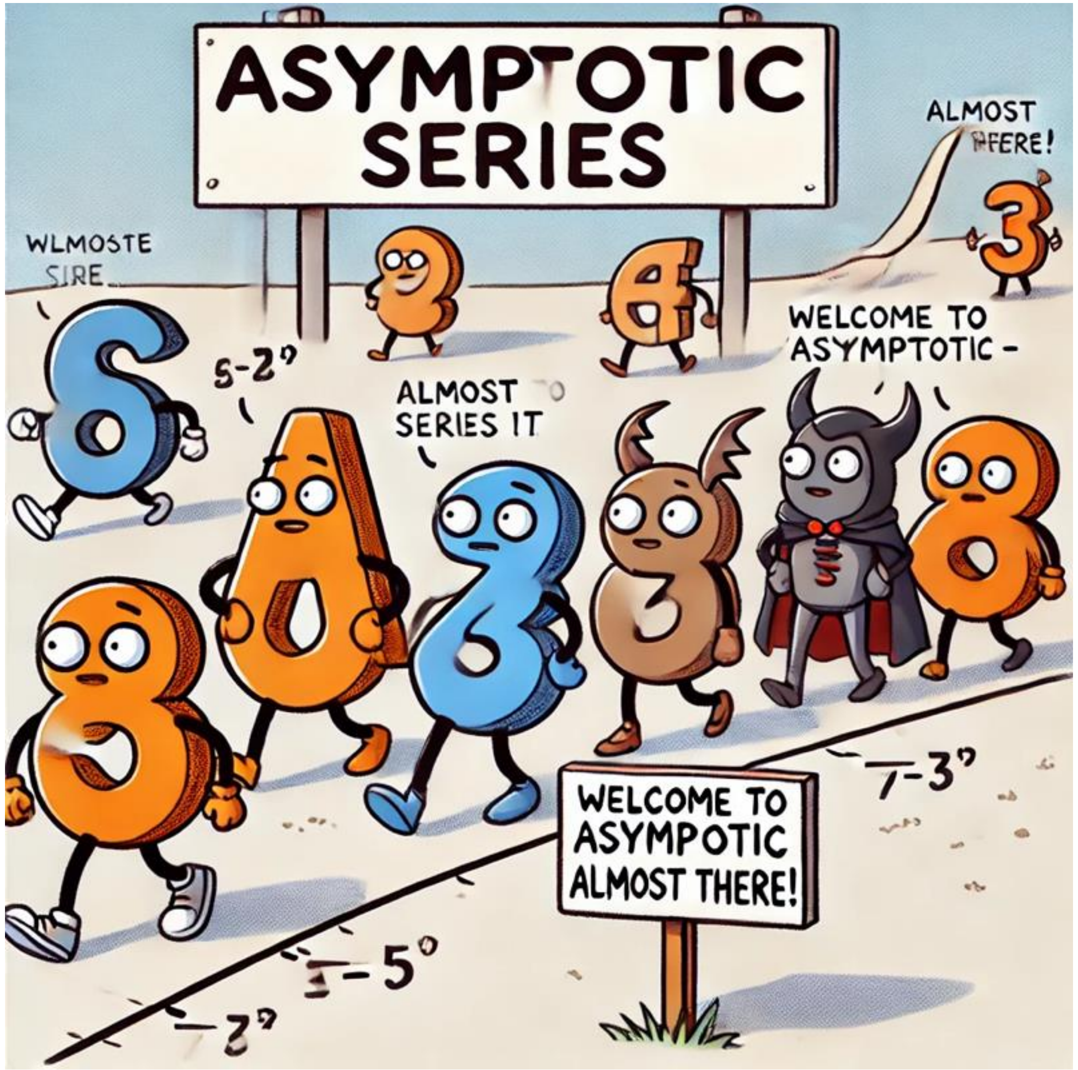
4. The Borel-Action Correspondence



5. Renormalons and instantons



1. Asymptotic Series

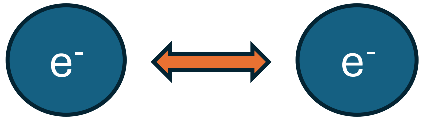


Perturbation theory **must** fail

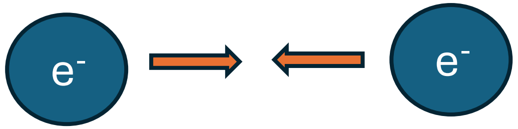
Divergence of Perturbation Theory in Quantum Electrodynamics

F. J. Dyson
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York
(Received November 5, 1951)

- If observable F(α) is analytic in α, then α > 0 and α < 0 would be similar at small α



α>0 repulsive



α<0 attractive

Very different!

Conclusion: Perturbation series $F(\alpha) = \sum_{n=0}^{\infty} a_n \alpha^n$

have zero radius of convergence

ex. 1: Quark pole mass

Beneke, Eur. Phys. J. Spec. Top., 2021

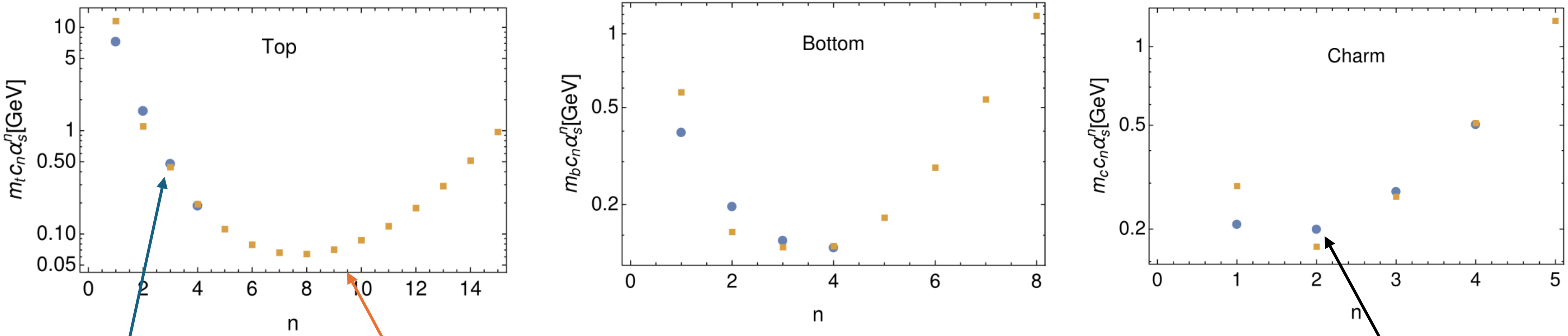
Quark pole masses known to 4-loops in QCD

$$m_t = 163.643 + 7.531 + 1.606 + 0.494 + 0.194$$

$$m_b = 4.200 + 0.400 + 0.199 + 0.145 + \mathbf{0.135}$$

$$m_c = 1.280 + 0.211 + \mathbf{0.202} + 0.282 + 0.510$$

$$m_{\rm pole} - m_{\overline{\rm MS}}(\mu) = \frac{C_F e^{5/6}}{\pi}\,\mu \sum_n (-2\beta_0)^n\, n!\, \alpha_s^{n+1}.$$



exact values

expected from inspired parameterization

starts to grow by n=2!

$$\tilde{c}^{(\rm as)}_{n+1} = (-2\beta_0)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)}\Bigg[1 + \frac{s_1}{n+b} + \frac{s_2}{(n+b)\,(n+b-1)} + \frac{s_3}{(n+b)\,(n+b-1)\,(n+b-2)} + \cdots\Bigg];$$

ex. 2: Non-global logs

MDS and Zhu arXiv:1403.4949

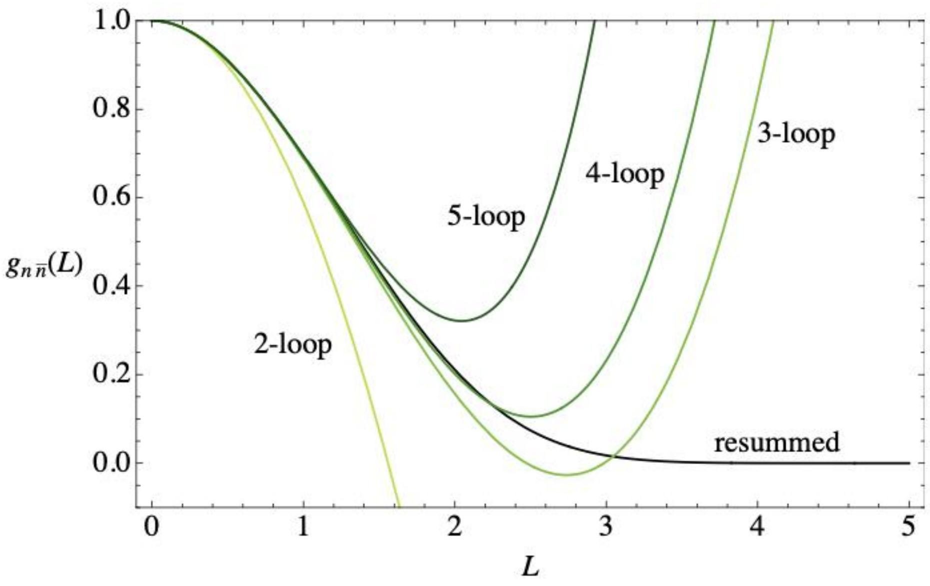
Series for leading non-global log in hemisphere mass distribution known to 5+ loops

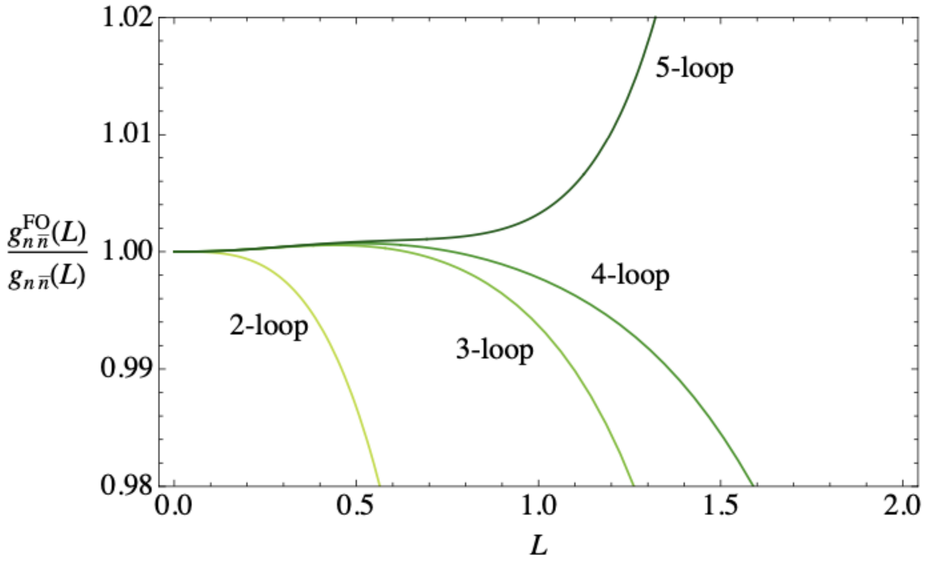
$$g_{n\bar{n}}(L) = 1 - \frac{\pi^2}{24}L^2 + \frac{\zeta(3)}{12}L^3 + \frac{\pi^4}{34560}L^4 + \left(-\frac{\pi^2\zeta(3)}{360} + \frac{17\zeta(5)}{480}\right)L^5 + \ldots$$

$$= 1 - 0.411233512L^2 + 0.10017141L^3 + 0.0028185501L^4 + 0.0037694522L^5 + \ldots$$

bigger

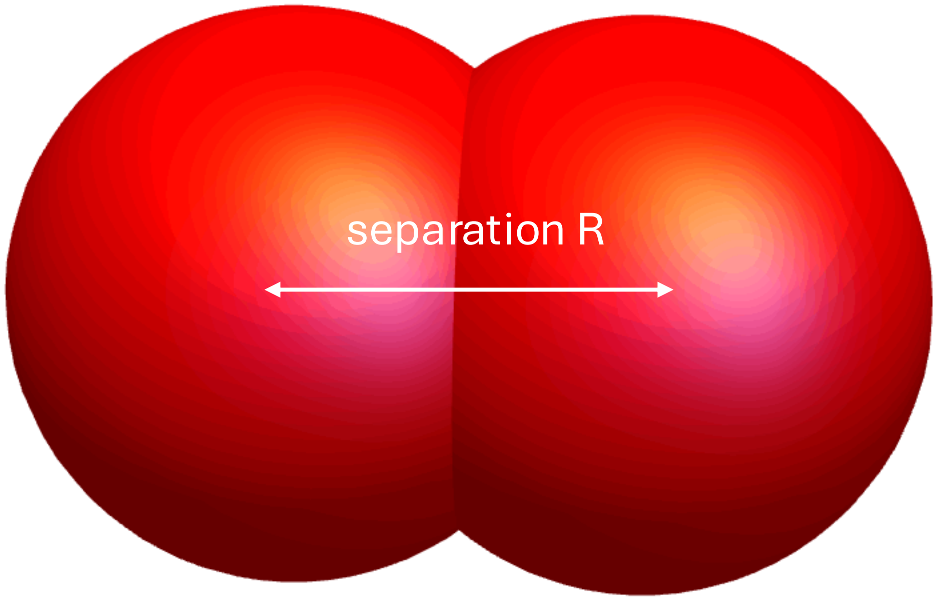
- Exact result known numerically (resummed)



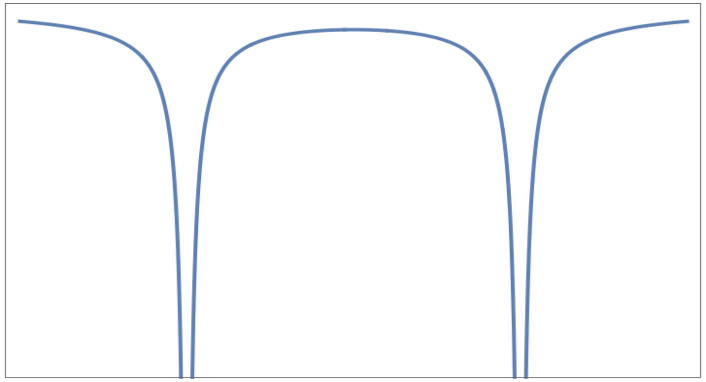


- Poor convergences suggests asymptotic series

ex. 3: Hydrogen molecule H_2^+



- R=∞, energy is that of isolated Hydrogen atoms



Electron is in a 3D double-well potential

- Compute energies as a series in 1/R

(calculation is in Landau and Lifshitz)

$$E_n \approx \sum_k \left(\frac{1}{R}\right)^k k!$$

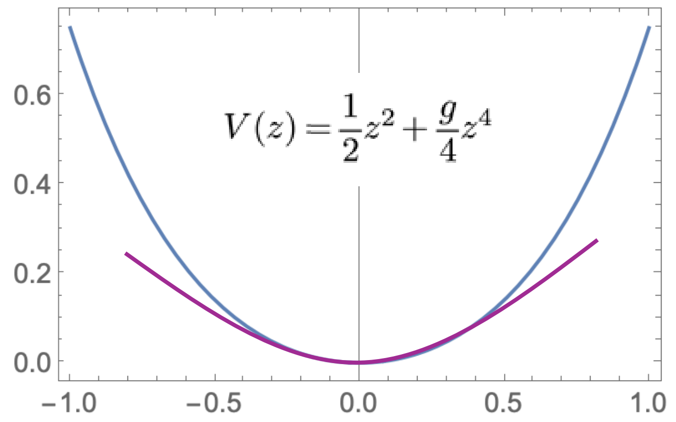
- coefficients grow factorially
- series has zero radius of convergence

1. Why do the coefficients grow factorially?

2. Why does perturbation theory work at all?

Two toy models

Anharmonic oscillator



Exact partition function

$$Z(g)=\int_{-\infty}^{\infty} dz\, e^{-\frac{1}{z}z^2+\frac{g}{4}z^4}$$

$$=\frac{1}{\sqrt{2g}}e^{\frac{1}{8g}}\mathcal{K}_{\frac{1}{4}}\left(\frac{1}{8g}\right)$$

Expand around minimum

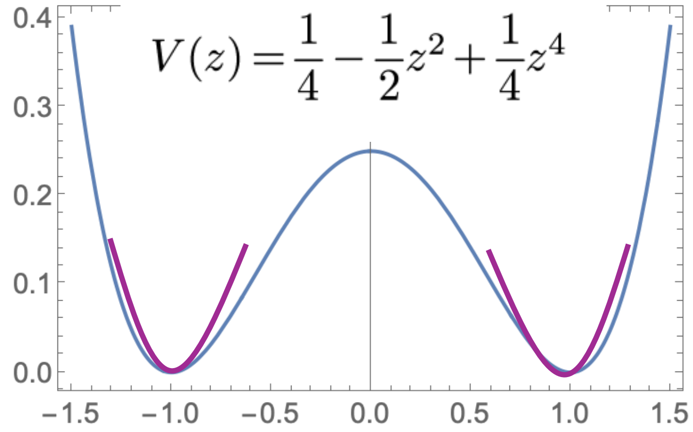
$$Z_N(g)=\sum_{n=0}\int_{-\infty}^{\infty} dz\, e^{-\frac{1}{2}z^2}\frac{1}{n!}\left(\frac{g}{4}z^4\right)^n$$

$$=\sum_{n=0}(-g)^n\sqrt{2}\frac{\Gamma\left(2n+\frac{1}{2}\right)}{n!}$$

$$\approx\sum_{n=0}\frac{1}{\sqrt{\pi}}(-4)^n n!\, g^n$$

asymptotic series
(alternating sign)

Double well



$$Z(g)=\int_{-\infty}^{\infty} dz\, e^{-\frac{1}{g}\left(\frac{1}{4}-\frac{1}{2}z^2+\frac{1}{4}z^4\right)}$$

$$=\frac{\pi}{2}e^{-\frac{1}{8g}}\left[\mathcal{I}_{-\frac{1}{4}}\left(\frac{1}{8g}\right)+\mathcal{I}_{\frac{1}{4}}\left(\frac{1}{8g}\right)\right]$$

Exact partition function

$$\approx\sqrt{g}\sum_{n}4^n n!$$

asymptotic series
(non-alternating sign)

1. Why factorial growth?

Anharmonic oscillator

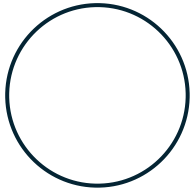
$$Z(g) = \int_{-\infty}^{\infty} dz\, e^{-\frac{1}{2}z^2 - \frac{g}{4!}z^4} \approx \sum_{n=0} \frac{1}{\sqrt{\pi}}(-4)^n n! g^n$$

- Partition function of a 0D quantum-mechanical system with action $S(z) = \frac{1}{2}z^2 + \frac{g}{4}z^4$
- Compute with Feynman diagrams

propagator = 1

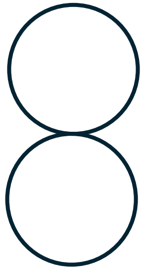
interaction = - g

symmetry factor



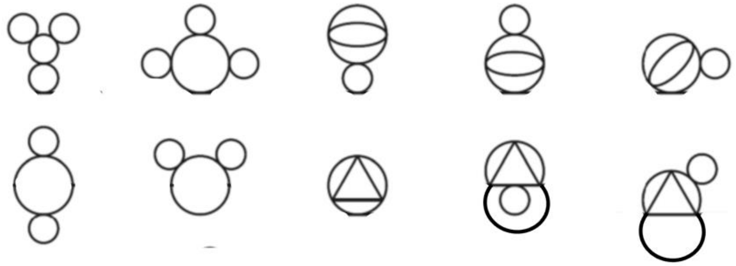
$=1 = \dfrac{Z(0)}{Z_0}$

$$Z_0(g) = \int_{-\infty}^{\infty} dz\, e^{-\frac{1}{2}z^2} = \sqrt{2\pi}$$



$= \;(-g) \times \dfrac{1}{8}$

$$\frac{1}{Z_0}\int_{-\infty}^{\infty} dz\, e^{-\frac{1}{2}z^2}\left(-\frac{g}{4!}z^4\right) = -\frac{g}{8}$$



- Sum of diagrams at order n is $(-4g)^n n!$
- Each diagram is < 1 (symmetry factor)

⇒ There must be at least n! diagrams at order n

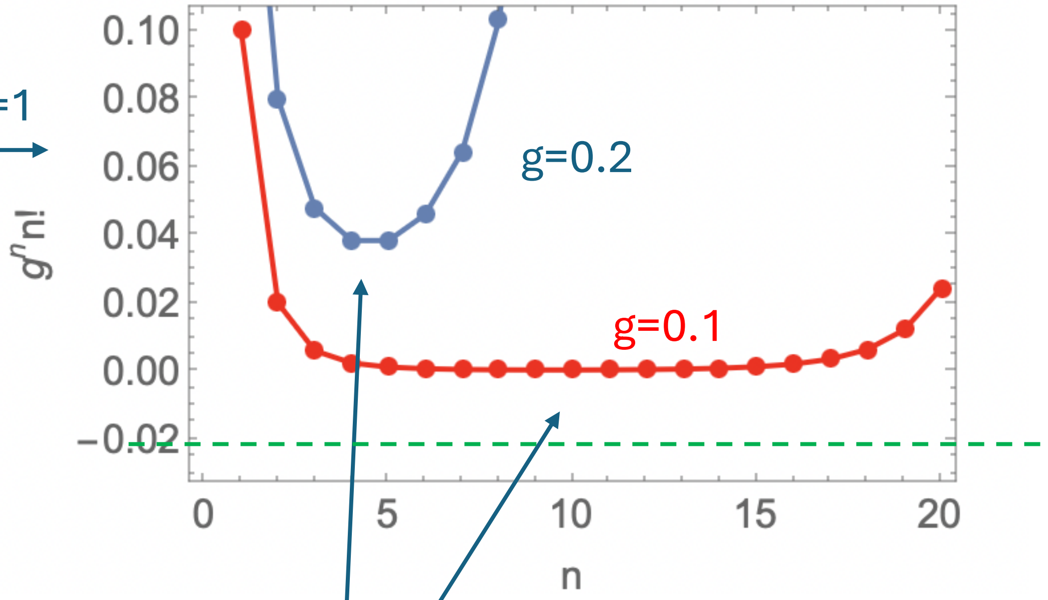
Factorial growth because there are n! diagrams at order g^n

Optimal truncation

2. Why does perturbation theory work?

$$Z(g) = \sum_n A^n g^n n!$$

A=1



Minimized when

$$\frac{\partial}{\partial n}(A^n g^n n!) \approx A^n g^n\, n! \ln(n\, A g\,) = 0 \quad \Rightarrow \quad n \approx \frac{1}{Ag}$$

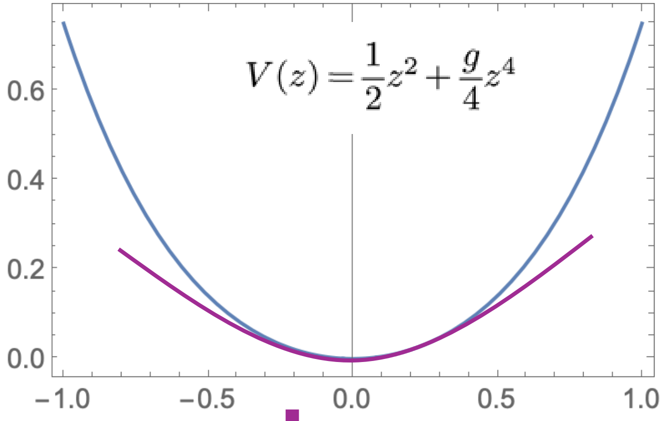
What is the right answer?

QED: $\alpha_e \sim \frac{1}{137}$ optimal truncation is 137 terms

QCD $C_A \alpha_s \sim 3 \times 0.118 \approx 0.33$ optimal truncation is 3 terms (NNLO)!

- Optimal trunction says there is a smallest term
- Explains why perturbation theory *seems* to work

Anharmonic oscillator



Expand around minimum

$$Z_N(g) = \sum_{n=0}^{N} \int_{-\infty}^{\infty} dz e^{-\frac{1}{2}z^2} \frac{1}{n!}\left(\frac{g}{4}z^4\right)^n$$

$$= \sum_{n=0}^{N} (-g)^n \sqrt{2} \frac{\Gamma(2n+\frac{1}{2})}{n!}$$

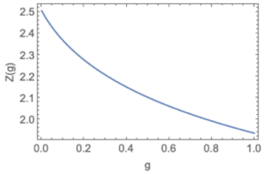
$$\approx \frac{1}{\sqrt{\pi}}(-4)^n n! g^n$$

asymptotic series
(alternating sign)

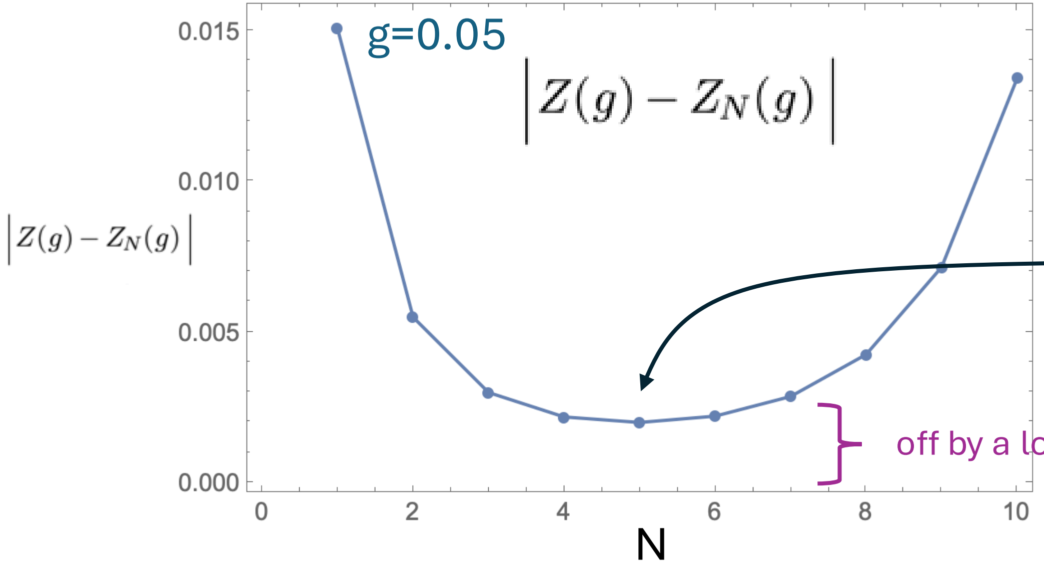
Compute exact answer

$$Z(g) = \int_{-\infty}^{\infty} dz e^{-\frac{1}{z}z^2 + \frac{g}{4}z^4} = \frac{1}{\sqrt{2g}} e^{\frac{1}{8g}} \mathcal{K}_{\frac{1}{4}}\left(\frac{1}{8g}\right)$$

smooth function



Compare expansion to exact result

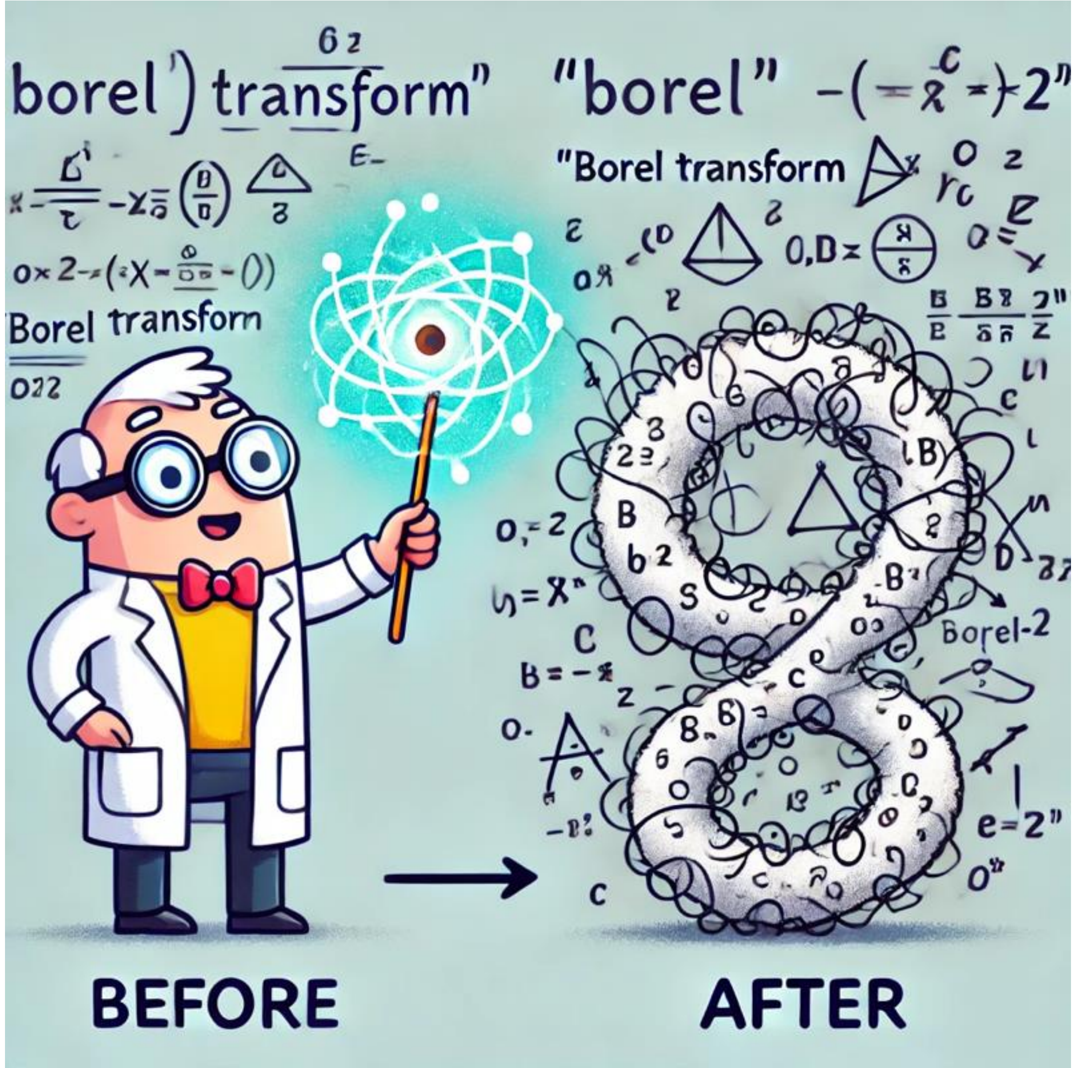


Optimal trunction:
N=4/g = 20

off by a lot!

3. Can we reconstruct the *exact* answer from the series?

2. Borel transform



Borel transform

Given a formal series $Z(g) = \sum_n a_n g^n$ its Borel transform is defined as $B(t) = \sum_n \frac{a_n}{n!} t^n$

Inverse Borel transform $\frac{1}{g}\int_0^\infty e^{-\frac{t}{g}} B(t) = \frac{1}{g}\int_0^\infty e^{-\frac{t}{g}} \frac{a_n}{n!} t^n = a_n g^n$ reproduces the original series

Typical asymptotic series

$$Z(g) = \sum_{n=0}^{\infty} A^n g^n n! \longleftrightarrow B(t) = \sum_{n=0}^{\infty} (At)^n = \frac{1}{1 - At}$$

pole at t=1/A

If Z is defined from an action

$$Z(g) = \int dz e^{-\frac{S(z)}{g}} = \int dt e^{-\frac{t}{g}} \frac{1}{|S'(z)|}$$

$$B(t) \propto \frac{1}{|S'(z)|}$$

A = coefficient of factorial growth
= singularity in Borel transform
= semiclassical pesuodparticle
= instanton

- B(t) has poles where S'(z) =0
- Poles are semi-classical objects: instantons

$\lambda\phi^4$ theory

Euclidean path integral

$$Z(\lambda) = \mathcal{N}\int \mathcal{D}\phi\, e^{-\int d^4x\left[\frac{1}{2}\phi\Box\phi + \frac{\lambda}{4}\phi^4\right]}$$

$$= \mathcal{N}'\int \mathcal{D}\phi\, e^{-\frac{1}{\lambda}\int d^4x\left[\frac{1}{2}\phi\Box\phi + \frac{1}{4}\phi^4\right]}$$

$$\approx \sum_{n=0}^{\infty}\left(\frac{3}{8\pi^2}\right)^n \lambda^n n!$$

asymptotic series

Asymptotic behavior associated with paths

$$\phi(x) = z\,\phi_b(x)$$

between classical solutions $\phi=0$ and $\phi=\phi_b$

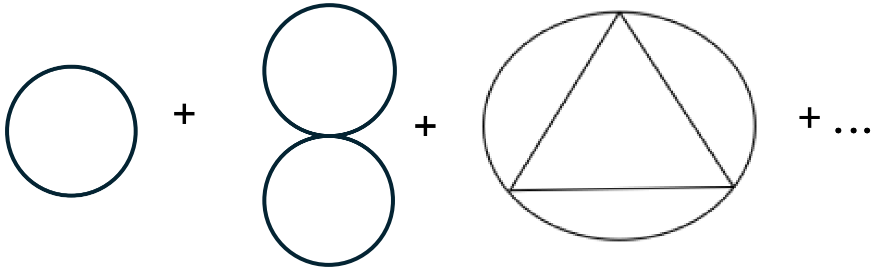
There is a non-trivial solution to equations of motion

$$\phi_b(x) = \frac{\sqrt{8}R}{R^2 + x_\mu x^\mu}$$

Fubini-Lipatov instanton

- Instanton action is $S_b = S[\phi_b] = \frac{8\pi^2}{3}$
- Consider one direction through field space $\phi(x) = z\,\phi_b(x)$

$$Z(\lambda) = \mathcal{N}'(\cdots)\int dz\, e^{-\frac{S_b}{\lambda}\left(\frac{1}{2}z^2 + \frac{1}{4}z^4\right)}$$

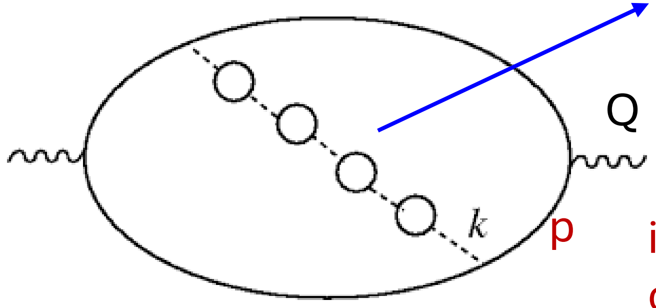


$$= \mathcal{N}'(\cdots)\sum_{n=0}^{\infty}\left(\frac{1}{S_b}\right)^n \lambda^n n!$$

Another source of factorial growth

Bubble chains in the photon propagator in QED

D(Q)=



each bubble gives

$$= \sum_{n=0}^{\infty} \alpha_s \int_0^{\infty} \frac{d\hat{k}^2}{\hat{k}^2} F(\hat{k}^2) \left[\beta_0 \alpha_s \ln\left(\frac{k^2 e^{-\frac{5}{3}}}{\mu^2} \right) \right]^n$$

integrate over p

integral over k gives n!

$$\int dk^2\, k^2\, (\beta_0 \alpha \ln k^2)^n = \left(\frac{-\beta_0}{2} \right)^n \alpha^n n!$$

Pole in Borel transform at

$$t = -\frac{2}{\beta_0}$$

- **Renormalon** = factorial growth associated with UV-divergent bubble chains
- Pole in Borel transform at t= n/β_0 for some integer n

4. Are renormalons also semi-classical objects?
- Does some field configuration have action S = n/β_0?

3. Can we reconstruct the *exact* answer from the series?

Borel resummation

The original function can be reconstructed via **Borel resummation**

$$B(t) = \sum_n \frac{a_n}{n!} t^n \qquad \frac{1}{g}\int_0^\infty e^{-\frac{t}{g}} B(t) = \frac{1}{g}\int_0^\infty e^{-\frac{t}{g}} \frac{a_n}{n!} t^n = a_n g^n$$

- reproduces the original series
- if series is convergent reproduces original function

e.g. anhamonic oscillator

$$Z(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-\frac{1}{g}\left(\frac{1}{2}z^2 + \frac{1}{4}z^4\right)}$$

$$= \frac{1}{\sqrt{2g\pi}} e^{-\frac{1}{8g}} \mathcal{K}_{\frac{1}{4}}\left(\frac{1}{8g}\right)$$

exact result

$$= \sum_{n=0}^{N} (-g)^n \sqrt{2} \frac{\Gamma(2n + \frac{1}{2})}{n!}$$

$$\approx \sqrt{g} \sum_{n=0}^{\infty} (-4g)^n n!$$

asymptotic series

Borel transform

$$B_1(t) = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2} + 2n)}{\Gamma\left(n + \frac{3}{2}\right) n!} \sqrt{2t}(-t)^n = 2\sqrt{\sqrt{1+4t} - 1}$$

Borel resummation

$$\frac{1}{g}\int_0^\infty e^{-\frac{t}{g}} 2\sqrt{\sqrt{1+4t} - 1} = \frac{1}{\sqrt{2g\pi}} e^{-\frac{1}{8g}} \mathcal{K}_{\frac{1}{4}}\left(\frac{1}{8g}\right) = Z(g)$$

Borel resummation reconstructs Z(g) exactly from its asymptotic series!

3b. Will Borel resummation always work?

Failure mode #1

$$e^{-\frac{1}{8g}}\mathcal{K}_{\frac{1}{4}}\left(\frac{1}{8g}\right)=\sum_n \frac{\Gamma(2n+\frac{1}{2})}{n!}g^n$$

$$e^{-\frac{6}{g}}+e^{-\frac{1}{8g}}\mathcal{K}_{\frac{1}{4}}\left(\frac{1}{8g}\right)=\sum_n \frac{\Gamma(2n+\frac{1}{2})}{n!}g^n$$

Two functions have same asymptotic series

series has no access to terms like this

Which will Borel resummation give?

Failure mode #2

e.g. double well

$$Z(g) = \int_{-\infty}^{\infty} dz\, e^{-\frac{1}{g}\left(\frac{1}{4}-\frac{1}{2}z^2+\frac{1}{4}z^4\right)} = \frac{\pi}{2} e^{-\frac{1}{8g}}\left[\mathcal{I}_{-\frac{1}{4}}\left(\frac{1}{8g}\right)+\mathcal{I}_{\frac{1}{4}}\left(\frac{1}{8g}\right)\right]$$

$$= \sum_{n=0}^{\infty} \sqrt{2g}\, g^n \frac{\Gamma\left(2n+\frac{1}{2}\right)}{n!}$$

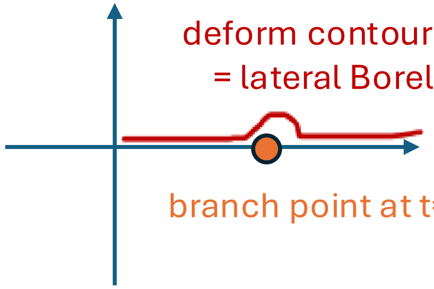
$$\approx \sqrt{g} \sum_{n=0}^{\infty} (4g)^n n!$$

non-alternating series

Borel transform

$$\mathcal{B}[f_2^{(1)}] = \sum_{n=0}^{\infty} t^{n+\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}+2n\right)}{n!\,\Gamma\left(\frac{3}{2}+n\right)} = \sqrt{2-2\sqrt{1-4t}}$$

- complex for t> ¼
- integral over 0<t< ∞ ambiguous



difference between deformations is

$$\frac{1}{g}\int_{C_+} e^{-\frac{t}{g}} B(t) - \frac{1}{g}\int_{C_-} e^{-\frac{t}{g}} B(t) = 2\pi i e^{-\frac{t^\star}{g}} B(t^\star)$$

Ambiguity $\sim \; i e^{-\frac{t^\star}{g}}$

What does Borel resummation do if B(t) has singularities?

Borel resummation of a *real series* of a *real function* is complex!

Trans-series

Series in quantum mechanics and quantum field theory are believed to be **trans-series**

$$Z(g) = \sum_{a,S_b} \ln^a g\, e^{-\frac{S_b}{g}} \left[\sum_n c_n^{a,b} g^n \right] \xrightarrow{\text{Borel resummation}} \sum_{a,S_b} \ln^a g\, e^{-\frac{S_b}{g}} f_n(g)$$

- each term can be complex
- imaginary parts cancel in the sum

Examples known to have trans series

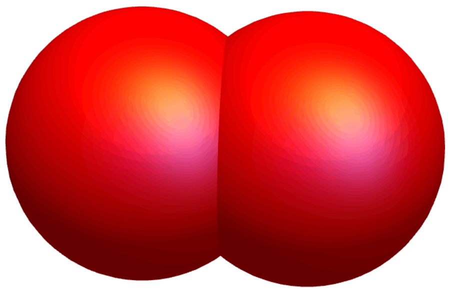
- energies in double well potential
- energies of H_2+ molecule
- correlation functions in 2D O(N) model, Gross-Neveu model

Examples believed to be trans-series

- quark pole masses
- e+e- event shapes
- any observable in QCD

Trans-series example #1

e.g. Energy levels of H_2+



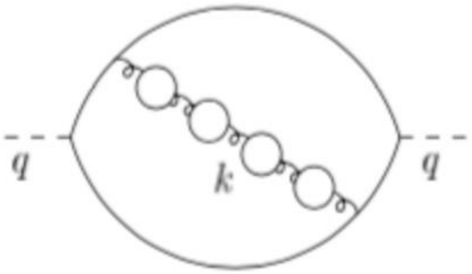
Damburg et al. PRL 52 13 (1984)

$$E(R) \sim \sum E^{(N)}(2R)^{-N} + e^{-R/n}\sum a^{(N)}(2R)^{-N}$$
$$+ e^{-2R/n}\left[\sum d^{(N)}(2R)^{-N} + \log R \ \text{terms}\right] \pm i e^{-2R/n}\sum c^{(N)}(2R)^{-N} + \dots,$$

- asymptotic series
- Borel resummation is complex. Imagninary part cancels against

Trans-series example #2

$J_\mu J_\nu$ = D =



$$= \left(\frac{-\beta_0}{2}\right)^n \alpha^n n! \longrightarrow \mathrm{Im}\, D = \pm i \pi e^{-\frac{1}{\beta_0 \alpha_s}}$$

Borel resummation

Operator Product Expansion $J_\mu J_\nu \sim (q_\mu q_\nu - q^2 g_{\mu\nu})\left[C_0 + \frac{C_2}{Q^4}\mathrm{Tr}\, G^2_{\mu\nu} + \frac{C_4}{Q^4}\bar{\psi}\psi \cdots\right]$

$$\alpha(Q) = \frac{1}{\beta_0 \ln\frac{Q^2}{\Lambda^2}}$$

$$\Rightarrow \Lambda^2 = e^{-\frac{1}{\alpha(Q)\beta_0}} Q^2$$

$$\Rightarrow \langle\Omega|\mathrm{Tr}\, G^2_{\mu\nu}|\Omega\rangle \approx \Lambda^4 = e^{-\frac{2}{\alpha_s(Q)\beta_0}} Q^4$$

$$\langle\Omega|J_\mu J_\nu|\Omega\rangle \sim (q_\mu q_\nu - q^2 g_{\mu\nu})\left[\left(\cdots + i\pi e^{-\frac{2}{\alpha_s \beta_0}}\right) + C_2 e^{-\frac{2}{\alpha_s(Q)\beta_0}} + \cdots\right]$$

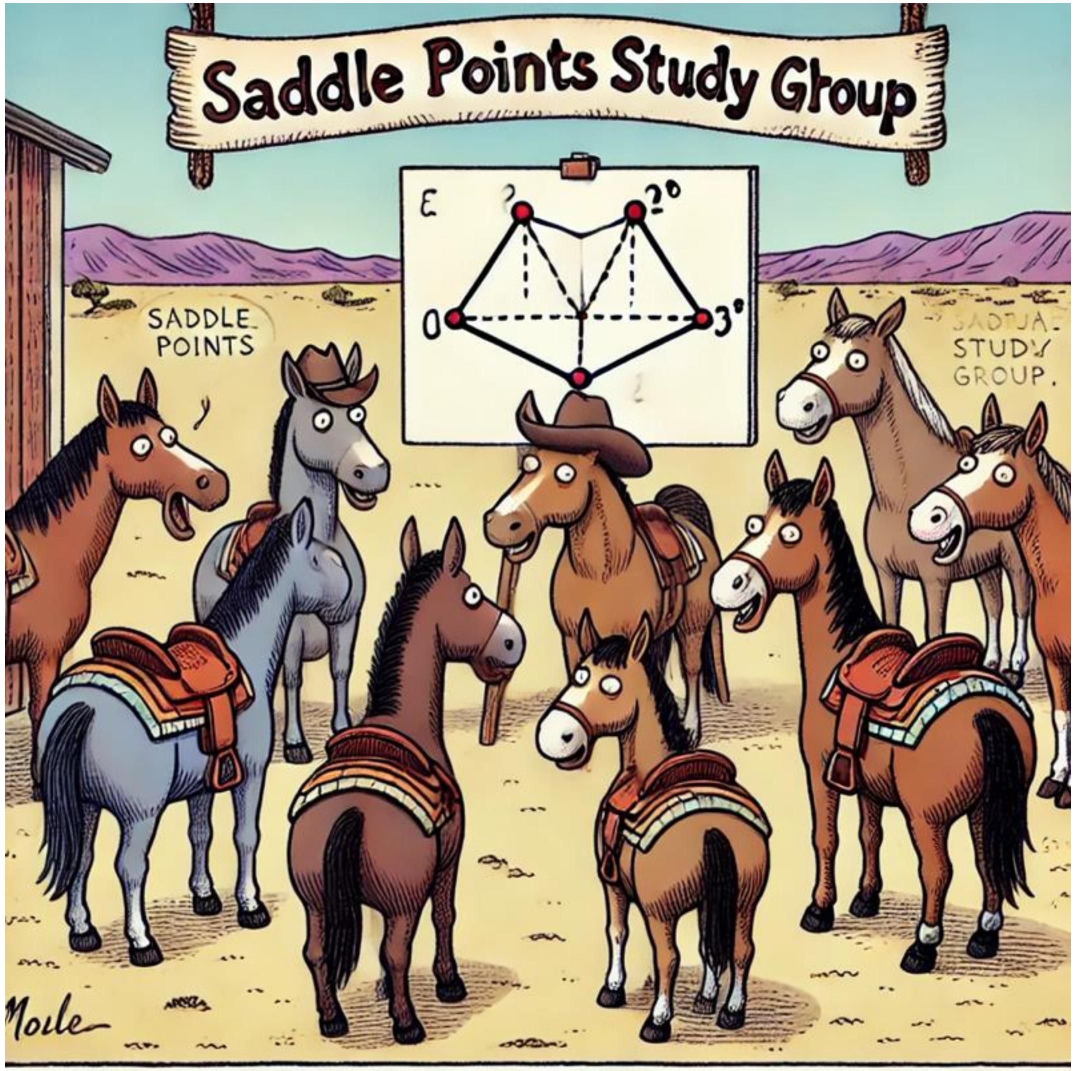
- $e^{-n/\beta_0 \alpha}$ Borel ambiguities suggest operators of dimension 2n
- Operator contributions cancel imaginary parts from Borel resummation

Why are operator expectation values complex?

Unanswered questions so far

1. Why does Borel resummation ever work?
2. When does Borel resummation reproduce a function?
3. What cancels the imaginary part when Borel transformation is ambiguous?
4. Is there a semi-classical interpretation of renormalons?

3. Saddle points



Saddle point approximation

Consider a 1D Laplace integral $Z(g) = \int dz e^{-\frac{S(z)}{g}}$

- We can expand around any point z* where S'(z) = 0

$$Z(g) = e^{-\frac{S(z^\star)}{g}} \int dz e^{-\frac{S''(z^\star)}{g}\frac{z^2}{2} - \frac{S'''(z^\star)}{g}\frac{z^3}{3!} + \cdots}$$

- rescale $z \longrightarrow \sqrt{g} z$
- expand petrubatively in g

$$= e^{-\frac{S(z^\star)}{g}} \sqrt{g} \int dz e^{-S''(z^\star)\frac{z^2}{2}} \frac{1}{n!}\left(\sqrt{g} S'''(z^\star)\frac{z^3}{3!} + \cdots\right)^n$$

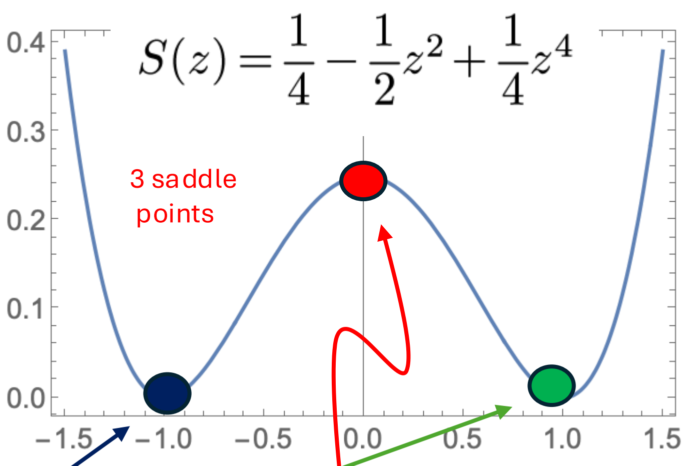
If S''(z*) > 0, integrate along *real* z
If S''(z*) < 0, must integrate along *imaginary* z

For a multidimensional Laplace integral $Z(g) = \int_C d^n z e^{-\frac{S(\vec{z})}{g}}$

- Can expand around any point where S'(z*) = 0 = **saddle point**
- Saddle point approximation requires integrating along direction where Re[S(z)] increases fastest

steepest ascent contours
= Lefshetz thimbles

Double well-example

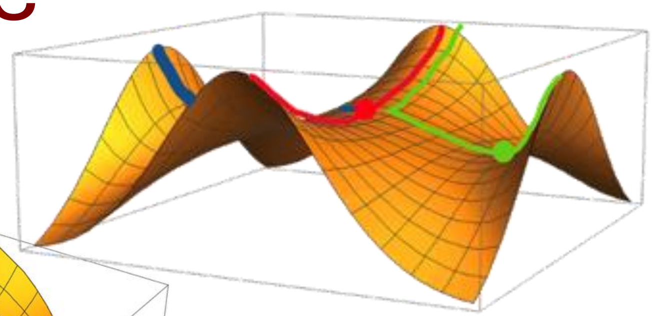


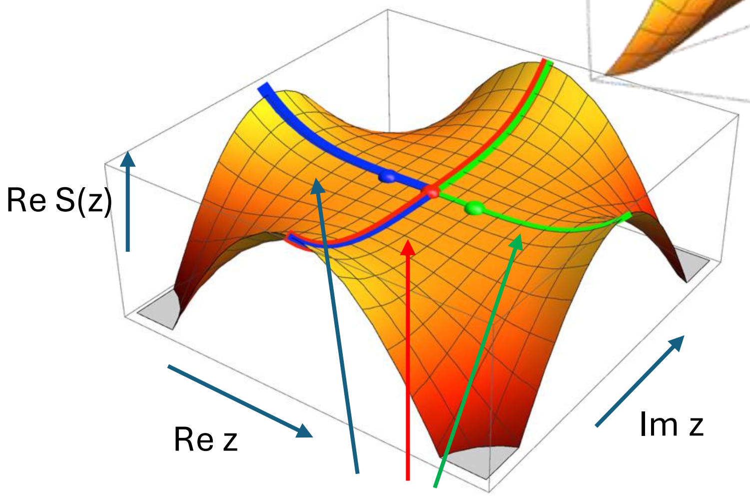
$z^*=-1$ or $z^*=1$

- S(z) increases in real z direction

$z^*=0$

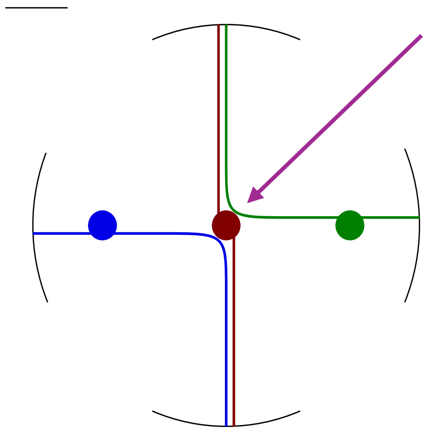
- S(z) decreases for real z
- S(z) increases for imaginary z





full complex steepest ascent trajectories are called **Lefshetz thimbles**

intersection of thimbles called a **Stokes point**



Theorem

The Borel resummation of the saddle point expansion around z* gives = the integral over the thimble passing through z*

Key to why Borel resummation works in physics

Picard-Lefshetz theory

Any contour can be decomposed into a sum over **thimbles** (uphill paths)

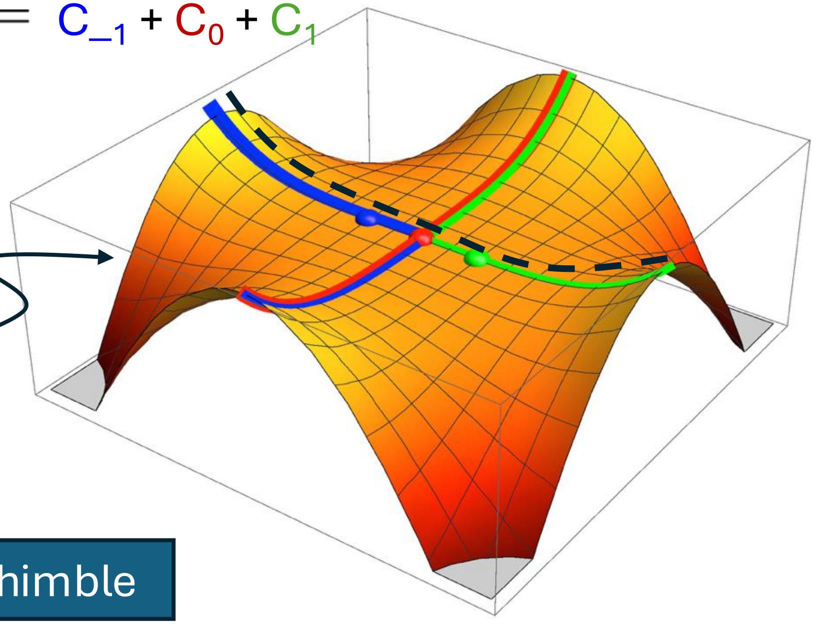
$$\mathcal{C} = \sum n_j \mathcal{C}_j$$

$$n_j = \langle \mathcal{K}_j, \mathcal{C} \rangle$$

intersection numbers between contour C and Lefshetz **anti-thimbles** (downhill paths)

black lines

$\mathbb{R} = C_{-1} + C_0 + C_1$

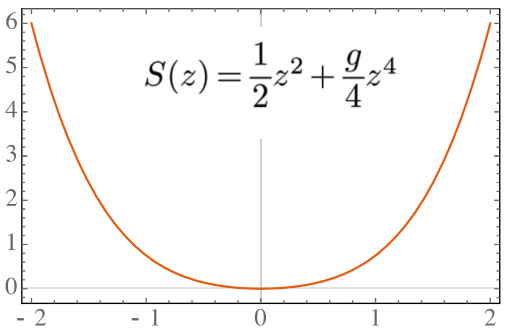


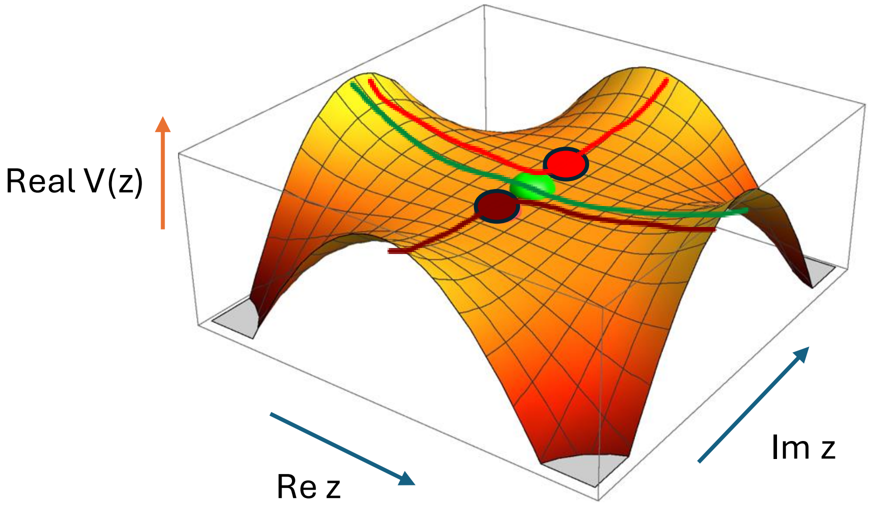
1. Borel resummation gives integral along thimble

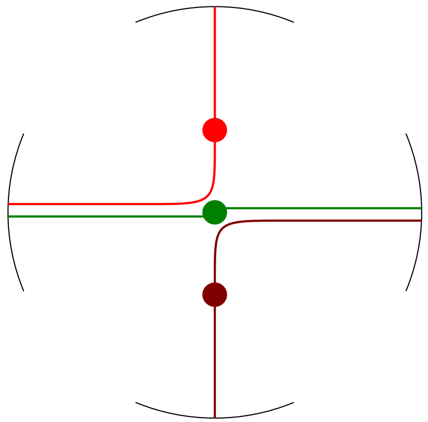
$$f(g) = \int_{\mathcal{C}} dz e^{-\frac{S(z)}{g}}$$

2. If integration contour C = a thimble
then Borel resummation will reconstruct f(g)

Anharmonic oscillator







Extrema at

$$z = 0, \frac{i}{\sqrt{g}}, -\frac{i}{\sqrt{g}}$$

- Steepest ascent contour passing through z=0 saddle is $\mathcal{C}_0 = \{z \in \mathbb{R}\}$
- Integral along C = R is already along a thimble

Just need a single contour (R)

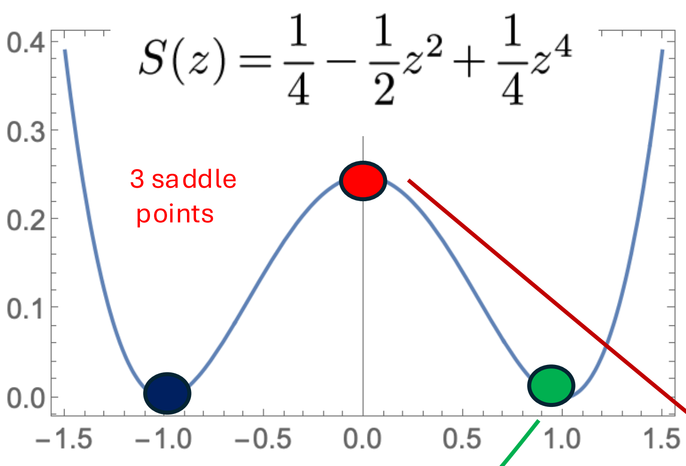
Borel transform

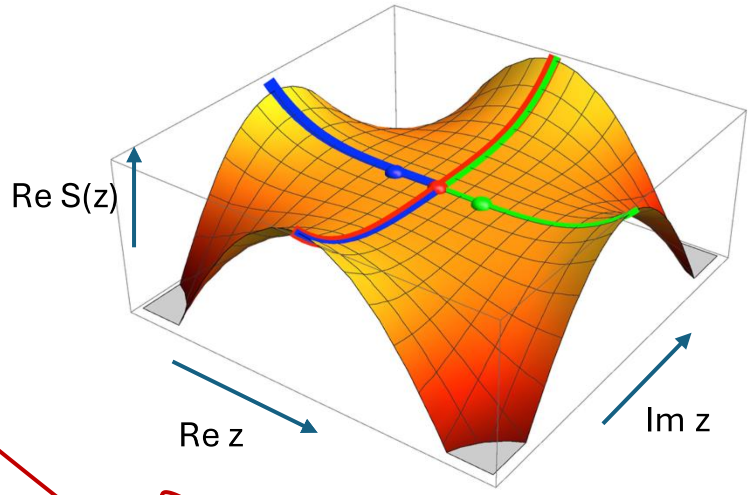
$$Z(g) = \int_{-\infty}^{\infty} dz e^{-\frac{1}{z}z^2 + \frac{g}{4}z^4} = \frac{1}{\sqrt{2g}} e^{\frac{1}{8g}} \mathcal{K}_{\frac{1}{4}}\left(\frac{1}{8g}\right) = \frac{1}{g}\int_0^{\infty} e^{-\frac{t}{g}} 2\sqrt{\sqrt{1+4t}-1}$$

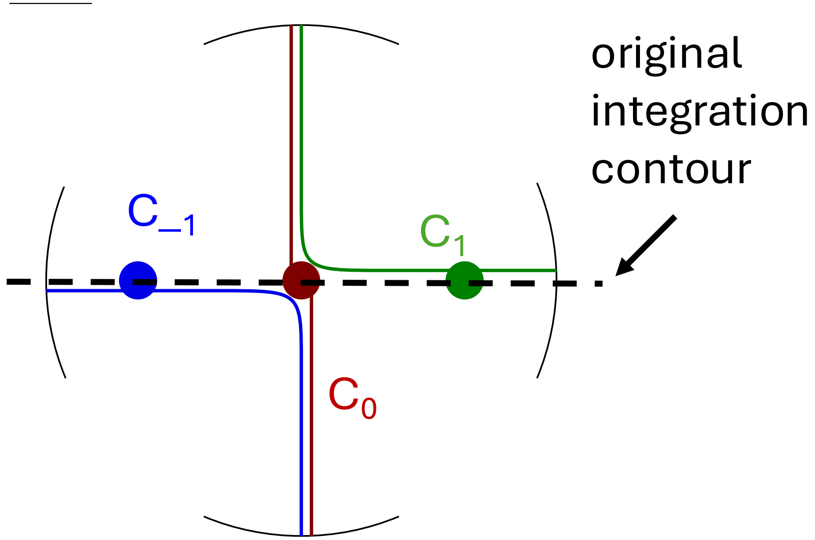
Borel resummation

We get the right answer from Borel resummation because thimble = original integration contour ✓

Double well-example







z=1 saddle

$$\mathcal{B}[f_2^{(1)}] = \sum_{n=0}^{\infty} t^{m+\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}+2n\right)}{n!\,\Gamma\left(\frac{3}{2}+n\right)} = \sqrt{2-2\sqrt{1-4t \pm i\epsilon}}\,.$$

branch of √ corresponds to which imaginary direction contour takes

Borel resummation

$$f^1(g) = \frac{1}{g}\int_0^{\infty} dt\, e^{-\frac{t}{g}} \sqrt{2-2\sqrt{1-4t\pm i\varepsilon}} = \frac{i}{2g} e^{-\frac{1}{8g}} K_{\frac{1}{4}}\left(-\frac{1}{8g} \pm i\varepsilon\right) = \int_{\mathcal{C}_1} dz\, e^{-S(z)}$$

Integral along thimble ✓

z=0 saddle

$$\mathcal{B}[f_2^{(0)}](t) = 2\sqrt{1-2\sqrt{t}\pm i\epsilon}\,\theta\left(t-\frac{1}{4}\right)$$

✓

Borel resummation

Integral along thimble

$$f^0(g) = \frac{1}{g}\int_{\frac{1}{4}}^{\infty} dt\, e^{-\frac{t}{g}} 2\sqrt{1-2\sqrt{t}\pm i\varepsilon} = \pm\frac{i}{\sqrt{2}} e^{-\frac{1}{8g}} K_{\frac{1}{4}}\left(\frac{1}{8g}\right) = \int_{\mathcal{C}_0} dz\, e^{-\frac{S(z)}{g}}$$

Sum of $f_2^0(g) + f_2^1(g) + f_2^{-1}(g) = \frac{\pi}{\sqrt{g}} e^{\frac{1}{8g}} \left[\mathcal{I}_{-\frac{1}{4}}\left(\frac{1}{8g}\right) + \mathcal{I}_{\frac{1}{4}}\left(\frac{1}{8g}\right)\right] = \int_{-\infty}^{\infty} dx\, e^{\frac{1}{2}x^2 - \frac{1}{4}gx^4}$ ✓

Summary of Picard-Lefshetz theory

- Path integrals are Laplace integrals $Z(g) = \int_C d^n z e^{-\frac{S(\vec{z})}{g}}$
- Perturbation theory comes from expanding around some z* where S'(z*) = 0
- Resulting **series** $Z(g) = \sum_n a_n g^n$ are **asymptotic**
- Any contour contour C can be **decomposed into thimbles**
- Borel resummation of the expansion around z* gives the integral along the thimble passing through z*
- If the original contour **C is a thimble**,
  - then **Borel resummation gives Z(g)**
- If the original contour **C is not a thimble**
  then Z(g) is the **sum of Borel resummations** of expansions around multiple saddles

Questions so far

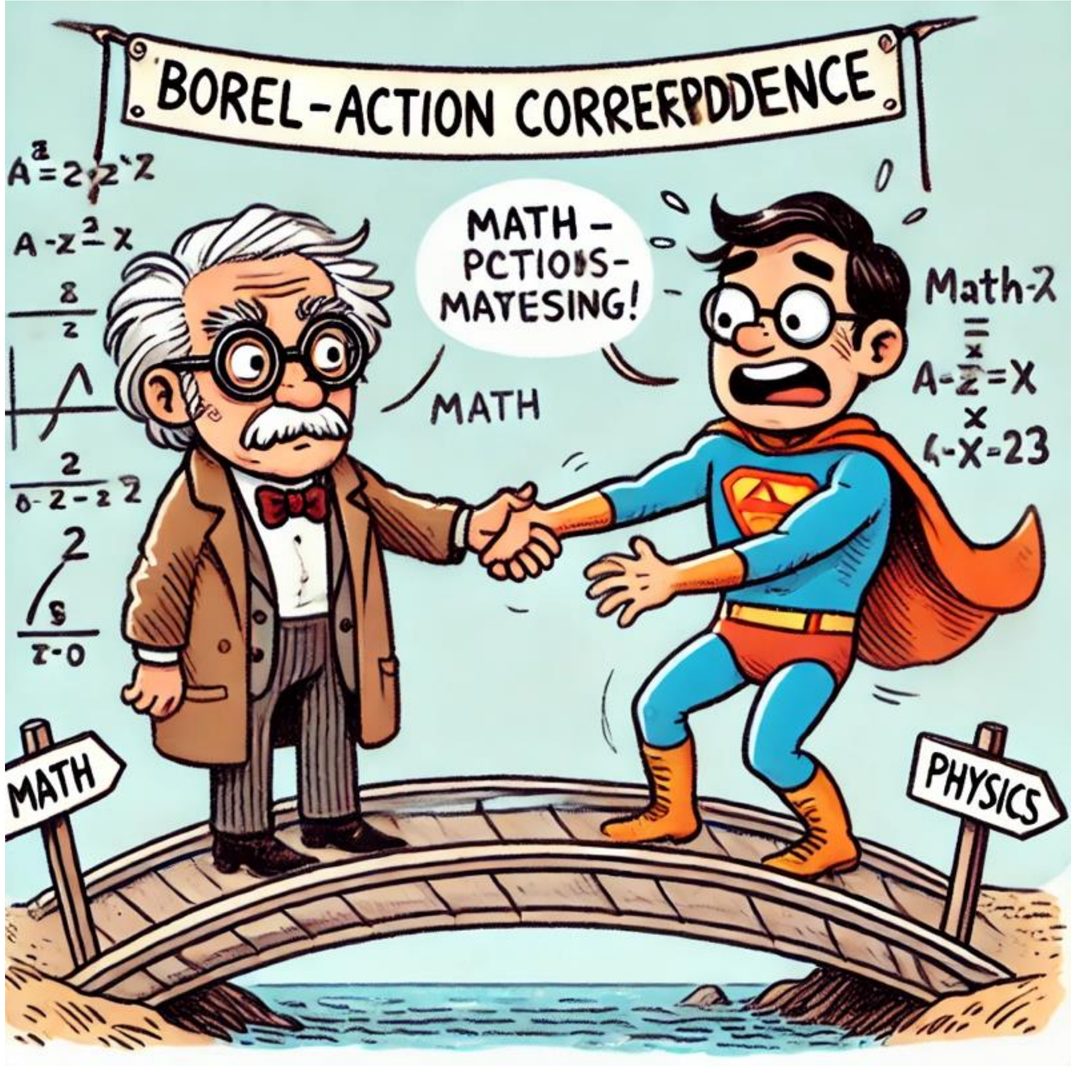
1. Why does Borel resummation ever work?
    - Because series come from saddle point expansions

2. When does Borel resummation reproduce a function?
    - If the integration contour is a thimble

3. What cancels the imaginary part when Borel transformation is ambiguous?
    - Integration along other contours

4. Is there a semi-classical interpretation of renormalons?

The Borel-action correspondence



Borel action correspondence

- We know: non-pertubative action S(z) → series $a_n g^n$ → Borel transform B(t)
- We want: perturbative series $a_n g^n$ → Borel transform B(t) → non-perturbative action S(z)

This can be done!

1 $$f(g) = \frac{1}{g}\int_0^\infty e^{-\frac{t}{g}} B(t) = \int_0^\infty e^{-\frac{t}{g}} B'(t) \Rightarrow \frac{1}{g} f(g) = \frac{1}{g}\int_0^\infty e^{-\frac{t}{g}} B'(t)$$

integration by parts

$$\Rightarrow \mathcal{B}\left[\frac{1}{g} f_S\right](t) = \frac{d}{dt}\mathcal{B}[f(g)]$$

action function S(z) = Borel variable t

2 $$\frac{1}{g} f(g) = \frac{1}{g}\int dz\, e^{-\frac{S(z)}{g}} = \frac{1}{g}\int dz\, e^{-\frac{t}{g}} \int dt\, \delta(t - S(z))$$

$$\Rightarrow \mathcal{B}\left[\frac{1}{g} f_S\right](t) = \int_{-\infty}^{\infty} dz\ \delta(t - S(z)) = \sum_{z_i | S(z_i) = t} \left|\frac{1}{S'(z_i)}\right|$$

$$\Rightarrow \frac{dB(S)}{dS} = \sum_{\text{domains}} \frac{1}{\left|\frac{dS}{dz}\right|} = \sum_{\text{domains}} \left|\frac{dz}{dS}\right| \Rightarrow B = \sum_{\text{domains}} |z_i|$$

S=t

Borel function B(t) = action variable z

- given B(t) can now invert to find S(z)

Borel action correspondence

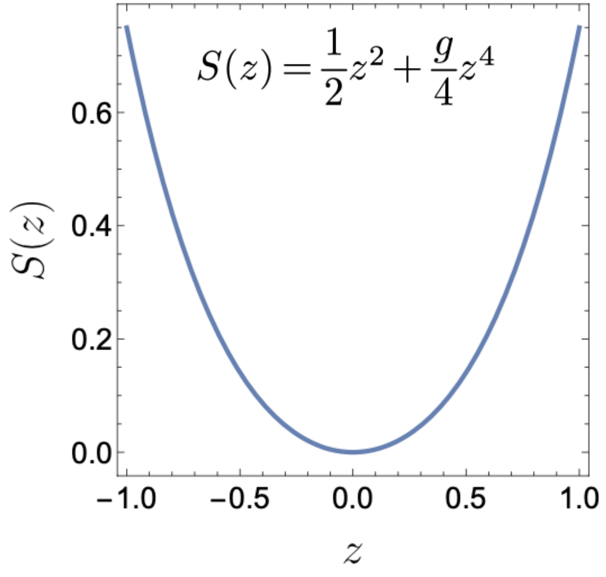
1. **Action function S(z) = Borel variable t**

2. **Borel function B(t) = action variable z**

3. **Stationary points of action = branch points of Borel transform**

Borel transform

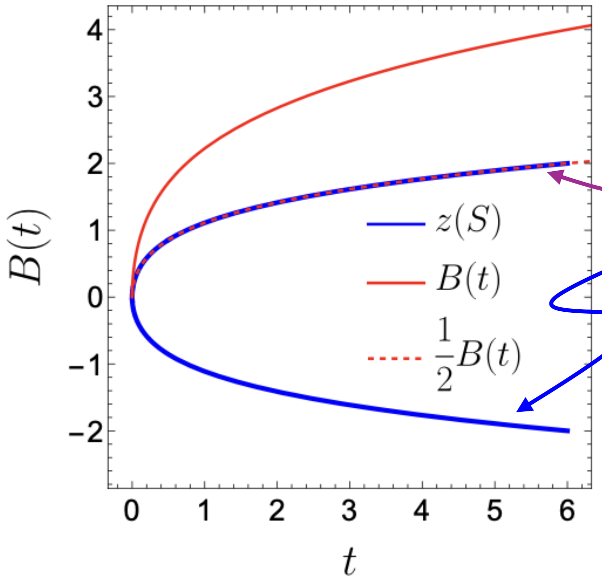
$$B(t) = 2\sqrt{\sqrt{1+4t}-1}$$



Solve S(z) = t

- two solutions

$$z_\pm = \pm\sqrt{\sqrt{1+4t}-1}$$



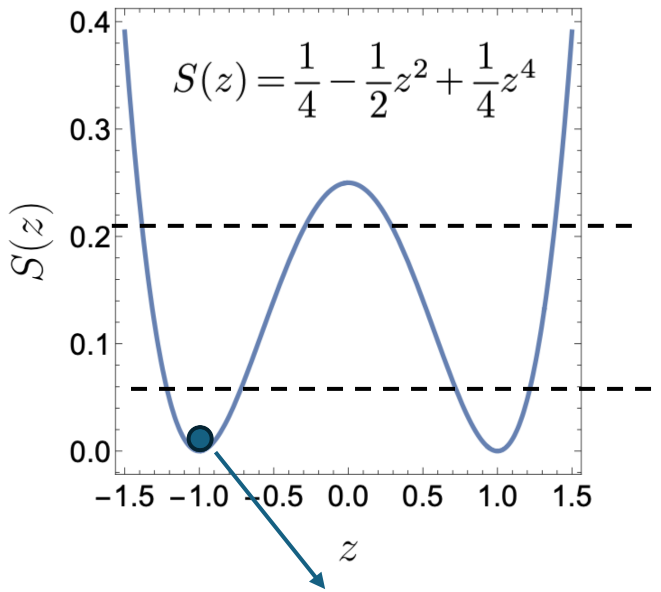
$$= |z_+| + |z_-|$$

Borel action correspondence

1. **Action function S(z) = Borel variable t**

2. **Borel function B(t) = action variable z**

3. **Stationary points of action = branch points of Borel transform**



Solve S(z) = t

- t > ¼: two solutions

$$z = \pm\sqrt{1+2\sqrt{t}}$$

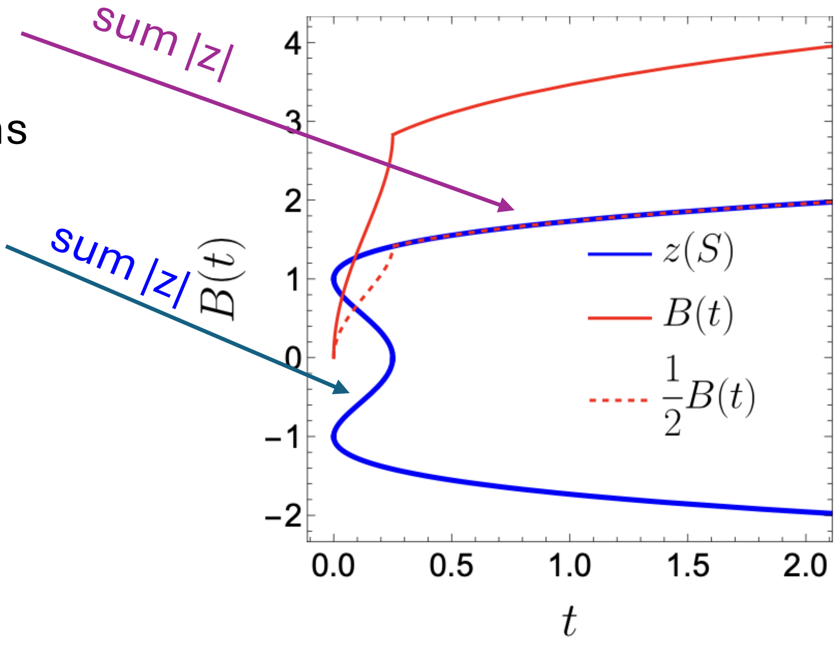
- t < ¼: four solutions

$$z = \pm\sqrt{1+2\sqrt{t}}$$

$$z = \pm\sqrt{1-2\sqrt{t}}$$

branch point at t = ¼ cancels

$$B(t) = -2\sqrt{1-2\sqrt{t}}\,\theta\left(\frac{1}{4}-t\right) + 2\sqrt{1+2\sqrt{t}}$$



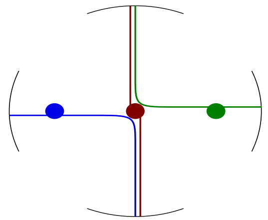
Borel transform of pert. series around saddle point

$$\mathcal{B}[f_2^{(1)}] = \sum_{n=0}^{\infty} t^{m+\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}+2n\right)}{n!\Gamma\left(\frac{3}{2}+n\right)} = \sqrt{2-2\sqrt{1-4t\pm i\epsilon}}\,.$$

branch point at = ¼

- indicates saddle point at S = ¼
- indicates new domain emerges

Stokes point:
thimbles intersect



Multidimensional version

1d version

$$\frac{d}{dt}\mathcal{B}[f(g)] = \int_{-\infty}^{\infty} dz\ \delta(t - S(z)) = \sum_{z_i | S(z_i)=t} \left| \frac{1}{S'(z_i)} \right|$$

multidimensional version

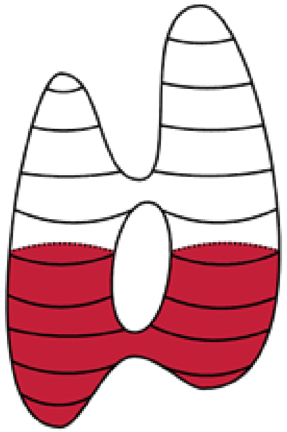
$$\frac{d}{dt}\mathcal{B}[f(g)] = \int d^n\vec{z}\ \delta(t - S(\vec{z})) = \int_{S(\vec{z})=t} d\sigma(\vec{z})\ \frac{1}{|\nabla S(\vec{z})|}$$

integrate in t

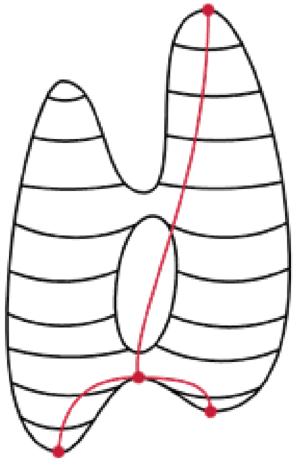
$$B(t) = \int d^n\vec{z}\ \Theta(t - S(\vec{z}))\,.$$

Borel transform
= cumulant density of states in field space
= volume of fields with action less than t

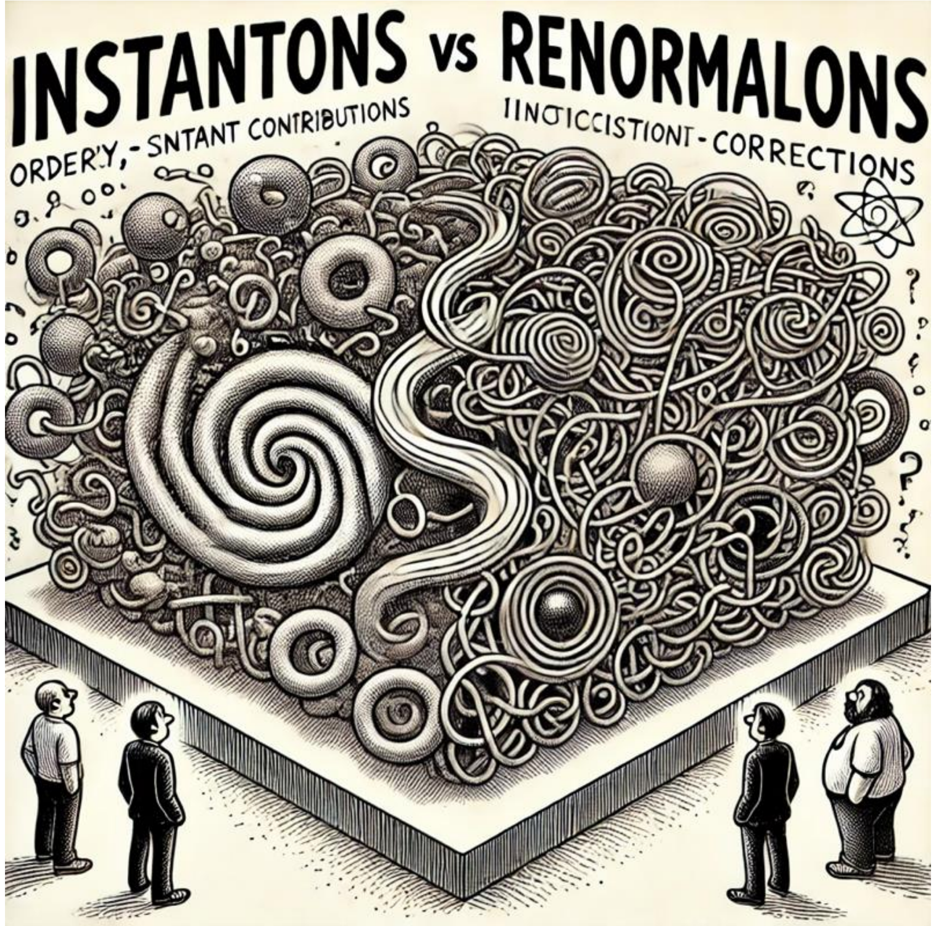
B(t) = coordinate volume with action S <t
=sublevel sets for **Morse** function S(z)



critical points of B(t) indicate toplogy change



5. Instantons and Renormalons

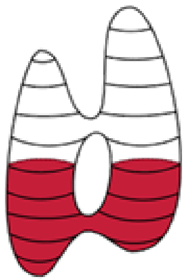


Multidimensional version

How are singularities of B(t) encoded in S(z)?

$$B(t) = \int d^n \vec{z}\, \Theta(t - S(\vec{z}))\,.$$

1 B(t) finite (branch point)
⇒ z = B finite



S(z) has a local extremum
- e.g. double well

B(t)= $\sqrt{2 - 2\sqrt{1 - 4t \pm i\epsilon}}\,.$

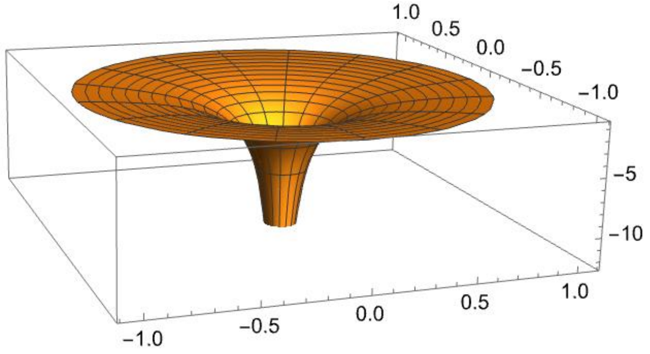
- e.g. 1d double-well
- Fubini instanton in $\lambda\phi^4$

2 Saddle at infinity
B(t) =∞ z =∞

e.g. $B(t) = -\ln(t^\star - t)$

Borel action correspondence
$\Rightarrow\ S(z) = t = t^\star - e^{-z}$

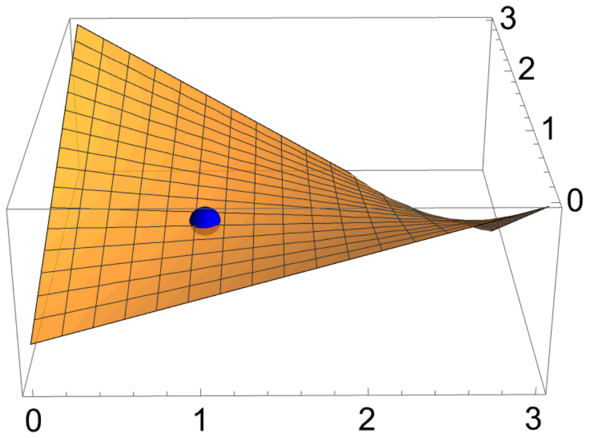
flat direction at infinity



- e.g. BPST instanton -anti-instanton pair in QCD

3 Saddle not at infinity
B(t) =∞ z =finite

- requires n>1
- action becomes unbounded at finite z



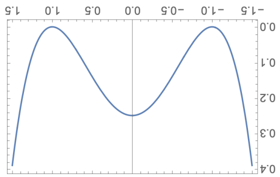
- renormalons

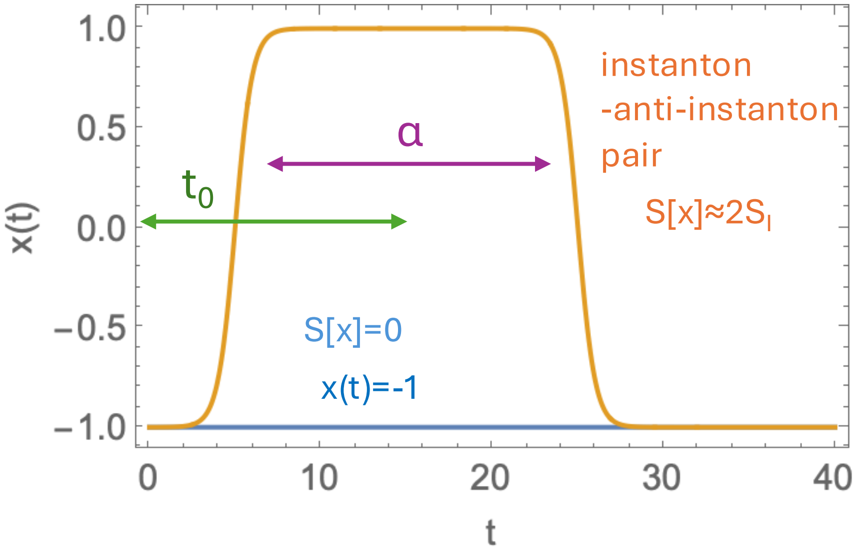
2 Saddles at infinity

Double well in quantum mechanics

$$\mathrm{Tr}[e^{-HT}] = \mathcal{N}\int \mathcal{D}x\, e^{-\frac{1}{g}\int dt\left(\frac{1}{2}\dot{x}^2 + V(x)\right)}$$

- Saddle points satisfy $\ddot{x} = V'(x)$
- Ball rolling down inverted potential
- Boundary conditions x(0)=x(T) = -1





BPST instantons in Yang Mills theory

- Need instanton-anti-instanton pair to have 0 toplogical charge

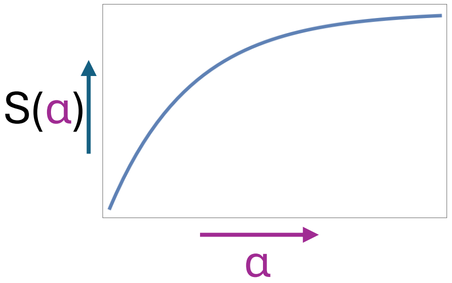
- exact solutions only at α=∞
- maps directly to double well

Babansky and Balitsky, PRL 85 (20) 2000

double instanton has two moduli t_0, α

$$x_{\mathbb{I}}(t) = x_I(t - t_0 + \alpha) - x_I(t - t_0 - \alpha)$$

$$S[x_{\mathbb{I}}] = 2S_I(1 - e^{-\alpha})$$

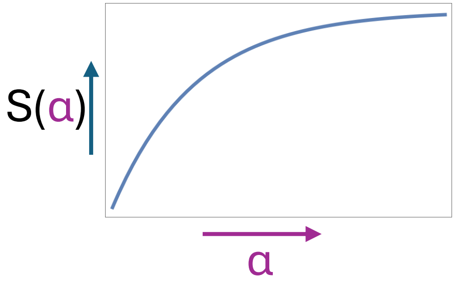


- action exactly independent of t_0 (on a circle)
- action independent of α at α = ∞

2 Saddles at infinity

- α is a quasi-collective coordiante

$$S(\alpha) = 2S_I(1-e^{-\alpha})$$

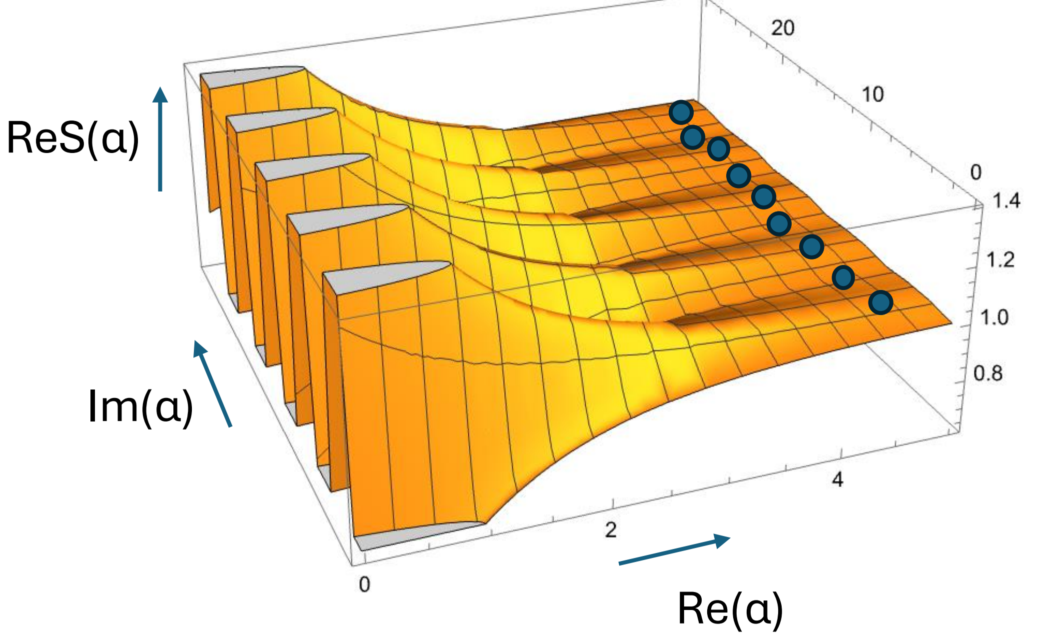


$$Z(g) = \int Dx\, e^{-\frac{S[x]}{g}} = (\cdots)\int d\alpha\, e^{-\frac{2S_I}{g}[1-e^{-\alpha}]}$$

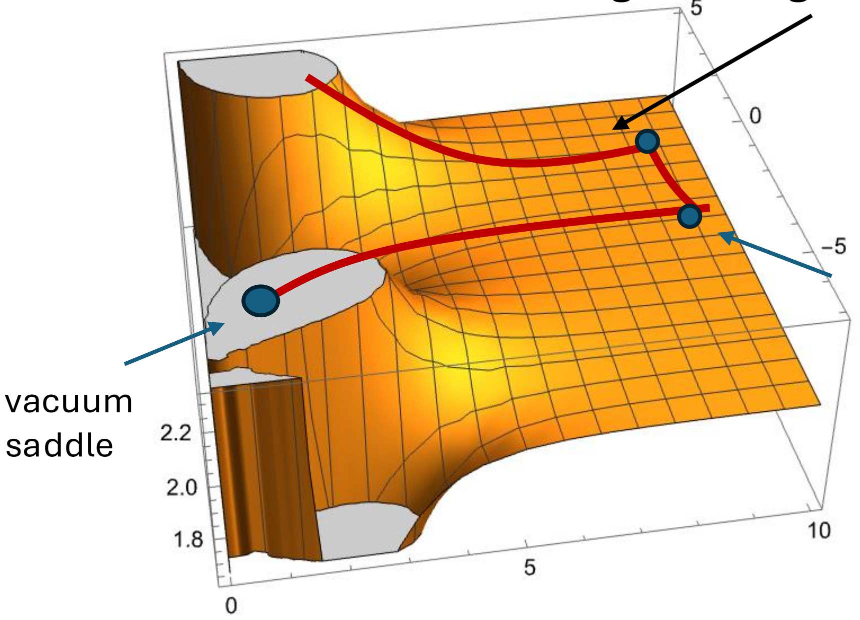
- integral over **α** is divergent
- α=∞ is not a maximum but a saddle
- additional saddles at $\alpha = \infty \pm n\pi i$

Behtesh et al.
arXiv1803.11533

$$S(\alpha) = 2S_I(1-e^{-\alpha})$$



integral along thimble gives



$$e^{-\frac{2S_I}{g}} E_1\left(-\frac{2S_I}{g} + i\varepsilon\right)$$

α=∞ saddle
=Stokes point

agrees with
trans-series for double well
(computed using exact WKB)

many Zinn-Justin papers
e.g. Zinn-Justin and Jentsschura
quant-ph/0501136

3 B= ∞ at z finite

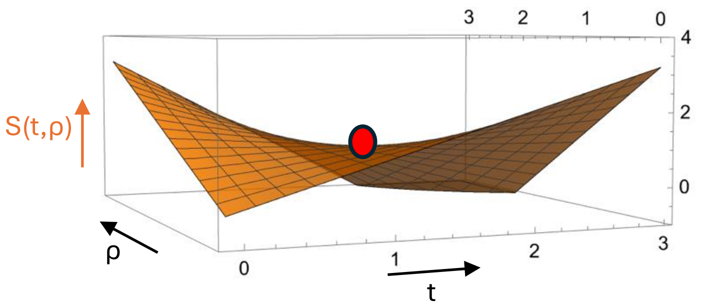
Example

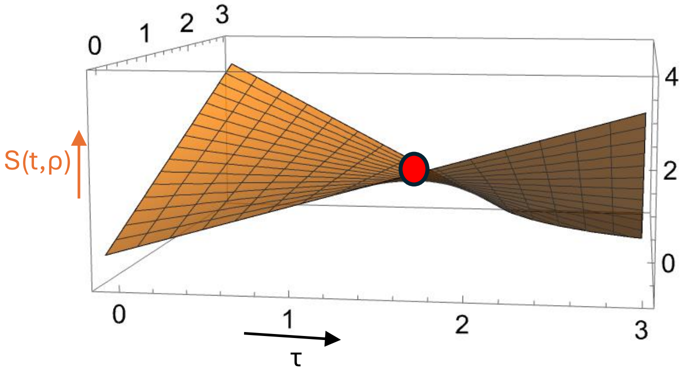
$$S(t,\rho) = \frac{t}{g} + k\rho - 2\beta t\rho$$

$$Z(g) = \int_0^\infty dt \int_0^\infty d\rho e^{-S(t,\rho)}$$ diverges

$$\int_0^\infty d\rho e^{-S(t,\rho)} = e^{-\frac{t}{g}} \frac{1}{k - 2t\beta}$$ if t < k/2β

Borel singularity at t=k/2β





S(t,ρ) increases in the ρ direction only if t < k/2β

- Integral over ρ is convergent if if t < k/2β
- Integral over r is divergent if if t > k/2β

Equations of motion

$$\partial_\rho S = k - 2t\beta \quad \Rightarrow \quad t^\star = \frac{k}{2\beta}$$

$$\partial_t S = \frac{1}{g} - 2\beta\rho \quad \Rightarrow \quad \rho^\star = \frac{1}{2g\beta}$$

saddle point is not at infinity

renormalon!

Renormalon saddles

arXiv:2410.07351
Bhattachary, Cotler, Dersy, MDS

operator of dimension k

scale-invariant action (e.g. QCD or $\lambda\phi^4$)

$$\langle \phi(0)^k \rangle = \int \mathcal{D}\phi e^{-\frac{1}{g(\mu)} S[\phi]} \phi(0)^k$$

arXiv:1707.08124
Andreassen, Frost, MDS

translation collective coordinate

dilatation collective coordinate

1. Choose a basis $\phi(x, R, x^0) = \sum_{n=0}^{\infty} a_n \frac{1}{R} \Phi_n\left(\frac{x - x_0}{R}\right)$

2. Integrate out UV modes (n>N) at 1-loop

$$\chi(x) = \sum_{n=0}^{N} a_n \Phi_n(x)$$

$$\langle \phi^k \rangle = \int d^4 x_0 \int \frac{dR}{R^{5+k}} \int d^N a_n e^{-\left(\frac{1}{g(\mu)} - 2\beta_0 \ln \mu R\right) S[\chi]} \mathcal{F}(a_n) \left[\chi\left(\frac{x_0}{R}\right)\right]^k$$

R dependence fixed by dimensional analysis

μ dependence fixed by RG invariance

Everything else from 1-loop integrals

3. Change to dimensionless ρ

$$\rho = \ln \mu R$$

$$\int \frac{dR}{R} \frac{1}{R^k} = \mu^k \int \frac{dR}{R} \frac{1}{(\mu R)^k} = \mu^k \int \frac{dR}{R} e^{-k \ln \mu R} = \mu^k \int d\rho e^{-k\rho}$$

$$S(t, \rho) = \frac{t}{g} + 2k\rho - \beta t \rho$$

4. Integrate over x_0

t =S

$$\langle \phi^k \rangle = \int d^N a_n \hat{\mathcal{F}}(a_n) \int d\rho e^{-\left(\frac{1}{g(\mu)} - 2\beta_0 \rho\right) S[\chi] - k\rho} = \int dt \int d\rho e^{-\left(\frac{1}{g(\mu)} - 2\beta_0 \rho\right) t - k\rho} \hat{B}(t)$$

Renormalon saddles

Generic QFT

$$\langle \phi^k \rangle = \int dt \int d\rho\, e^{-\left(\frac{1}{g(\mu)} - 2\beta_0 \rho\right)t - k\rho} \hat{B}(t)$$

nontrivial saddle at

$$t^\star = \frac{k}{2\beta_0} \qquad \rho^\star = \frac{1}{g(\mu)\beta_0}$$

k=4

$$t^\star = \frac{2}{\beta_0}$$

correct location

$\rho = \ln \mu R$

$$R^\star = \frac{1}{\mu} e^{\rho^\star} = \frac{1}{\mu} e^{\frac{1}{\alpha_s(\mu)\beta_0}} = \frac{1}{\Lambda_{\rm QCD}}$$

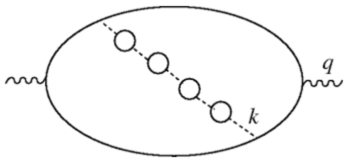
scale associated with renormalon is exactly $\Lambda_{\rm QCD}$

QCD

leading operator in OPE

$$\langle \Omega | \mathrm{Tr}\, G_{\mu\nu}^2 | \Omega \rangle \approx \Lambda^4 = e^{-\frac{2}{\alpha_s(Q)\beta_0}} Q^4$$

dimension k=4



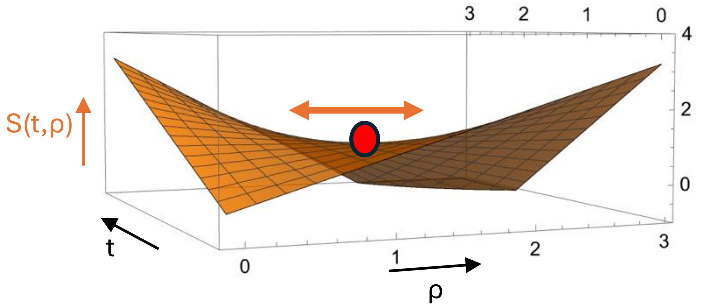
$$= \left(\frac{-\beta_0}{2}\right)^n \alpha^n n!$$

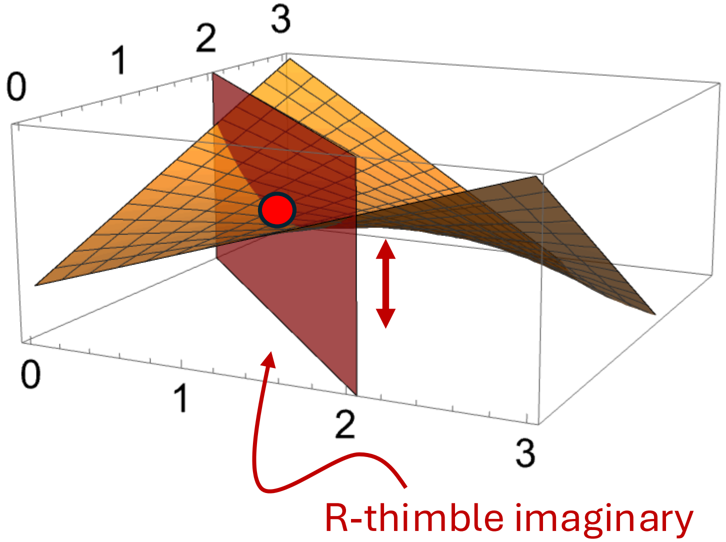
leading renormalon Borel pole at $t^\star = \frac{2}{\beta_0}$

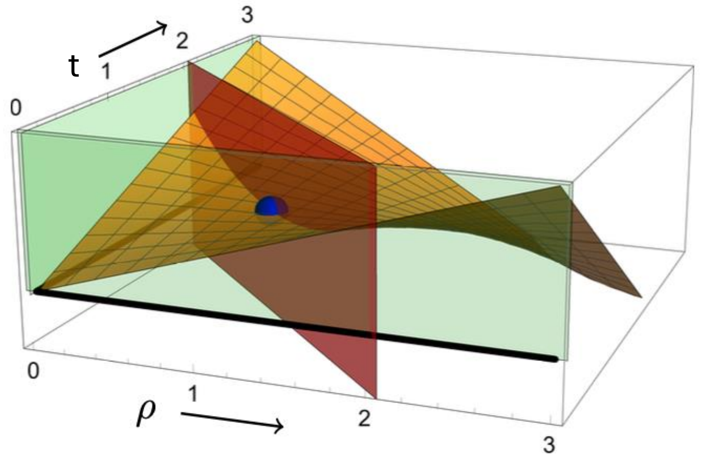
Renormalon Thimbles

trivial saddle

- starts on boundary
- bends complex at renormalon
- deformed to two half-imaginary semi-planes

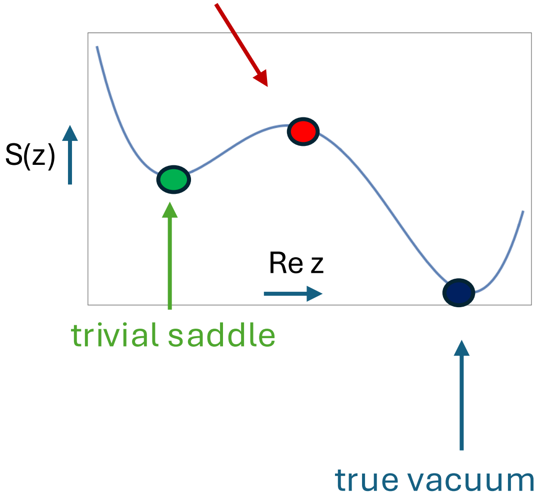


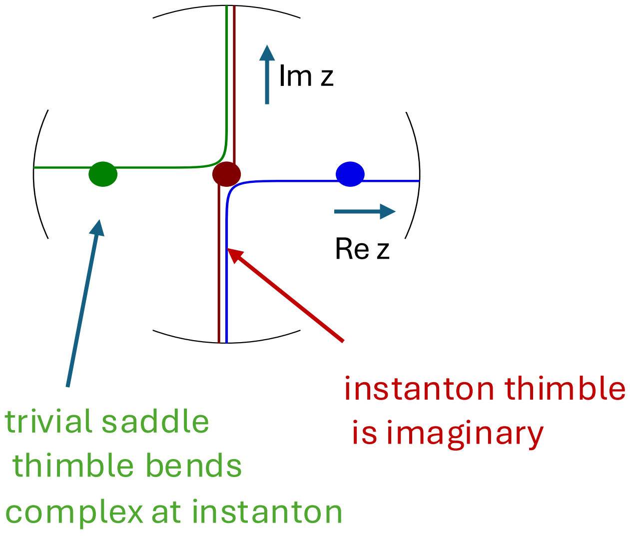




- Re S increases around renormalon
  in one real and one imaginary direction

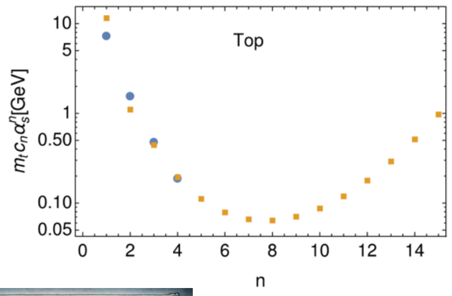
instanton in double well





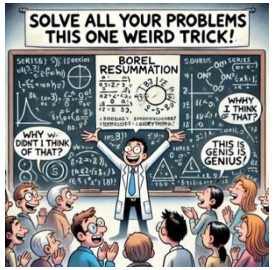
- instantons mediate tunneling in QM and QFT
- do renormalons mediate tunneling in QCD?
- is there a “dilute renormalon gas”?

Conclusions



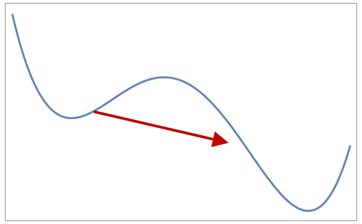
- Perturbation theory generically gives asymptotic series

$$f(g) \sim \sum_{n=0}^{\infty} A^n n! g^n$$



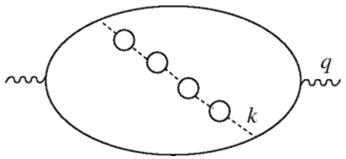
- Series cannot be summed, but may be Borel resummed

- Growth associated with instantons and renormalons

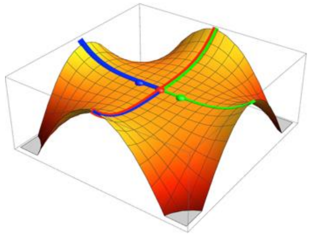


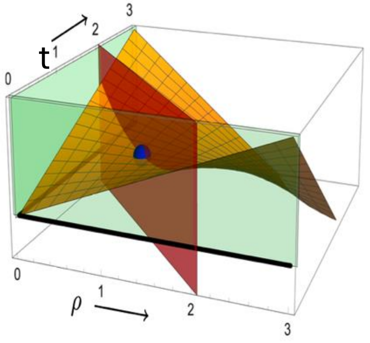
associated with tunneling

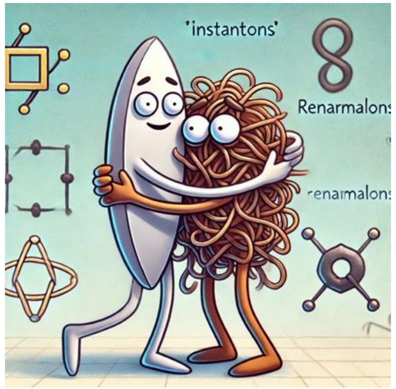
associated with running coupling



- Instantons and renormlaons can be unified







- New hope for connecting perturbative and non-perturbative physics in quantum field theory!