Landau, Cutkosky, and Pham: Geometry and Analyticity of Scattering Amplitudes

University of Chicago

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Based mostly on

arXiv:2211.07633 "Constraints on Sequential Discontinuities from the Geometry of On-shell Spaces"
Holmfridur S. Hannesdottir, Andrew J. McLeod, **MDS** and Cristian Vergu

and a bit on

arXiv:2007.13747 "Sequential Discontinuities of Feynman Integrals and the Monodromy Group"
J. Bourjaily, Holmfridur S. Hannesdottir, Andrew J. McLeod, **MDS** and Cristian Vergu

arXiv:1911.06821 "An S-matrix for massless particles" Holmfridur S. Hannesdottir and **MDS**

Outline

1. Introduction
   - How I got interested in this subject
   - Discontinuities, imaginary parts and monodromies
2. Landau equations
   - Geometric interpretation
3. Vanishing cells
   - Deforming integration contours
4. Constraints on sequential discontinuities
   - Tangential vs transversal intersections
5. Conclusions

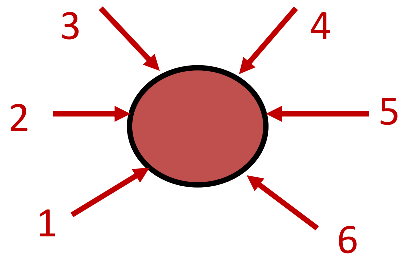
The S-matrix

- Is the S matrix completely fixed by physical constraints (causality, analyticity, etc.)?
  - Key question of the 1960s
  - Quantum Field Theory was shown capable of explaining strong interactions
    - S matrix program on hold for 40 years
- Recent progress in perurbation theory has renewed interested in analytic structure
  - More “data” – explicit calculations
  - **Mathematics** of functions appearing in amplitudes (cluster algebras, etc.)
    - Very efficient ways to write down amplitudes,
  - Success in the perturbative S-matrix bootstrap
    - collinear limits, Regge limits, conformal invariance, Steinmann relations
    - N=4 SYM 6 point amplitude bootstrapped to 7 loops [Caron-Huot et al 1903.10890]

Steinman relations are constraints on sequntial discontinuities [Steinmann 1960]

possible term: $\ln(p_1+p_2)^2 \ln(p_3+p_4)^2$

not allowed (at any order): $\ln(p_1+p_2+p_3)^2 \ln(p_2+p_3+p_4)^2$

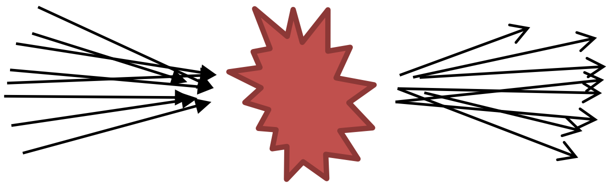


How can we understand constraints like this?

How did I get into this?

The S matrix describes the scattering of particles

$|\psi_{\rm in}\rangle$ "In state"



"out state" $|\psi_{\rm out}\rangle$

How is the S-matrix actually defined?

$$S \stackrel{?}{=} \lim_{t\to\infty} e^{-iHt}$$

- Doesn't exist: infinitiely oscillating phase

$$S \stackrel{?}{=} \lim_{t\to\infty} e^{iH_0 t} e^{-iHt}$$

[Wheeler, Heisenberg 1960]

- Works for mass-gapped theory
- Infrared divergent in gauge theories

$$S \stackrel{?}{=} \lim_{t\to\infty} e^{iH_A t} e^{-iHt}$$

- H_A is the "asymptotic Hamiltonian"
    - Includes all long-range interactions, e.g. Coulomb phase

[Dollard 1970, Fadeev and Kulish 1970]

How did I get into this?

What about QCD or N=4 SYM theory?

- N=4 is supposed to be a beautiful simple theory with lots of symmetry
- **Why should an S matrix that doesn't exist have any symmetry?**

"Remainder functions" have nice properties: $R_n = \ln\left[\frac{\mathcal{M}_n}{\mathcal{M}_n^{\mathrm{BDS}}}\right]$

[Bern, Dixon, Smirnov 2005]

BDS Anzatz: $\mathcal{M}_n^{\mathrm{BDS}} = \exp\left[\sum_L \left((4\pi e^{-\gamma})^\epsilon \frac{g_s^2 N_c}{8\pi^2}\right)^L \left(f^{(L)}(\epsilon) M_n^{(1)}(L\epsilon) + C^{(L)} + E_n^{(L)}(\epsilon)\right)\right]$

- R_n respects dual conformal invariance but violate Steinmann relations

BDS-like ansatz [Alday, Giotto, Maldacena 2009]

- violates dual conformal invariance but respects Steinmann relations

[Hannesdottir and MDS 2020]

Taking $H_A = H_{SCET}$ gives a finite S matrix for QCD and N=4

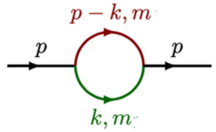
$$S = \lim_{t\to\infty} e^{iH_{\mathrm{SCET}}t} e^{-iHt}$$

- S matrix elements are finite and agree with BDS-like remainder functions
  - Unifies coherent/dressed states, SCET, and modern amplitude calculations

What properties does the finite S matrix have?

Example

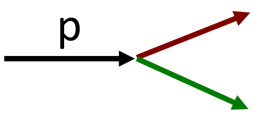
Consider the simplest 1-loop diagram: the bubble in d=2

$I_{\bigcirc}(p) =$ 

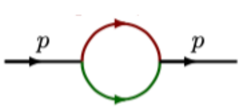
$$= \int d^2k \frac{1}{k^2 - m^2 + i\varepsilon} \frac{1}{(p-k)^2 - m^2 + i\varepsilon} = \frac{-2\pi}{\sqrt{s\,(s-4m^2)}} \ln \frac{\sqrt{4m^2 - s} - i\sqrt{s}}{\sqrt{4m^2 - s} + i\sqrt{s}}$$

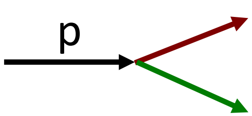
Even this diagram is remarkably rich, as we will see.

- At has a **normal threshold** branch cut starting at $s=4m^2$
  - For $s > 4m^2$ the on-shell process $p \to p_1 + p_2$ is allowed for physical on-shell momenta



  - Tree-level process tells you about singularities of loop amplitudes
  - e.g., through optical theorem

Im 

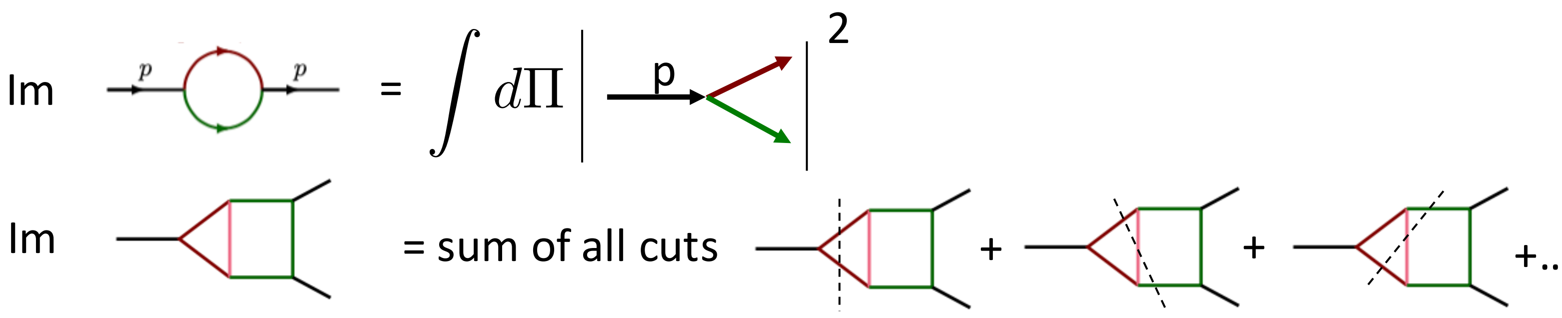
$= \int d\Pi \,\Big|$ 

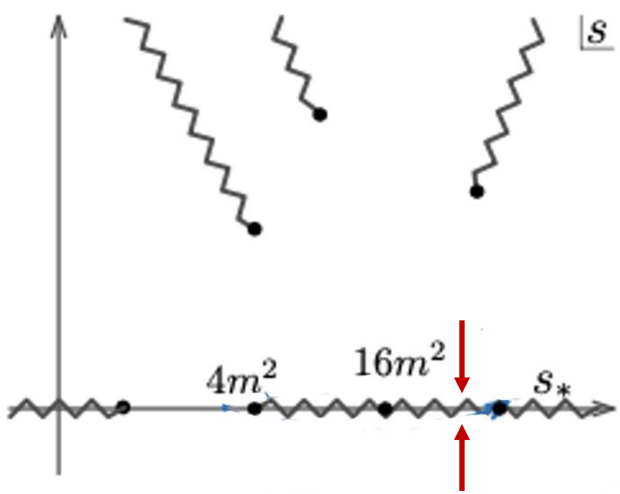
$\Big|^2$

- Not singular at the **pseudthreshold** s=0
  - There is a branch point at s=0 accessible with complex momenta
  - Does not correspond to anything physical happening

Imaginary part is too blunt

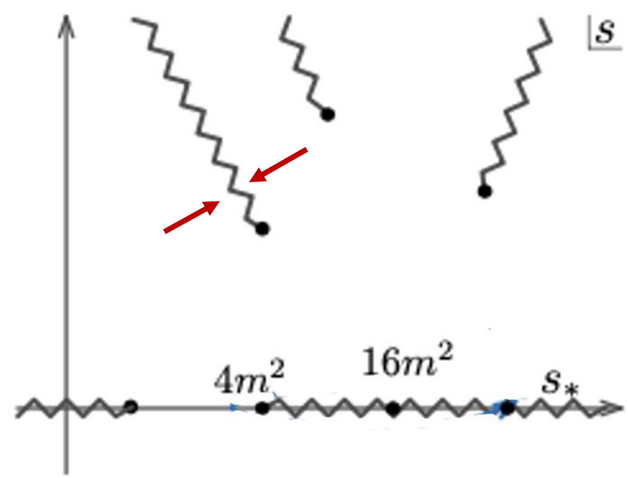
Optical theorem





Imaginary part gives the total discontinuity

- Cannot distinguish overlapping branch cuts

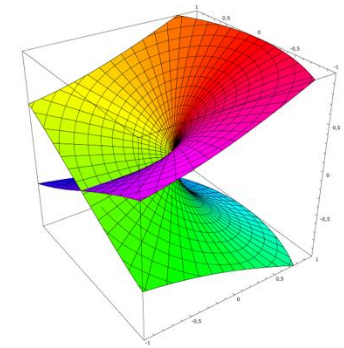


- To understand full analytic structure need to isolate each branch point/cut

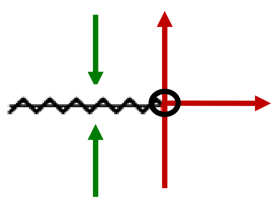
Branch points/cuts

Square root $\sqrt{x}$

- Single valued on Riemann surface



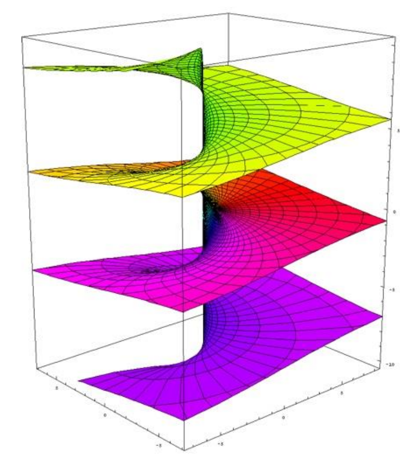
- Not singular at x=0
- Sign ambiguity: $\sqrt{-x} = \pm i\sqrt{|x|}$
- Branch cut is projection of Riemann surface onto complex plane
- Discontinuity gives back the same function



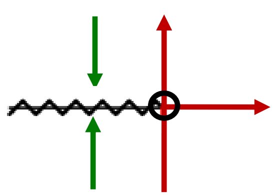
$$\text{Disc}\,\sqrt{x} = 2\sqrt{x}\,\theta(-x)$$

Logarithm: ln x

- Singular at x=0
- Phase ambiguity on negative real axis $\ln(-x) = \ln x \pm \pi i$
- Riemann surface is infinite sheeted
- Discontinuity gives back a simpler function



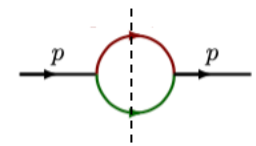
$$\text{Disc}\ln(x) = 2\pi i\,\theta(-x)$$



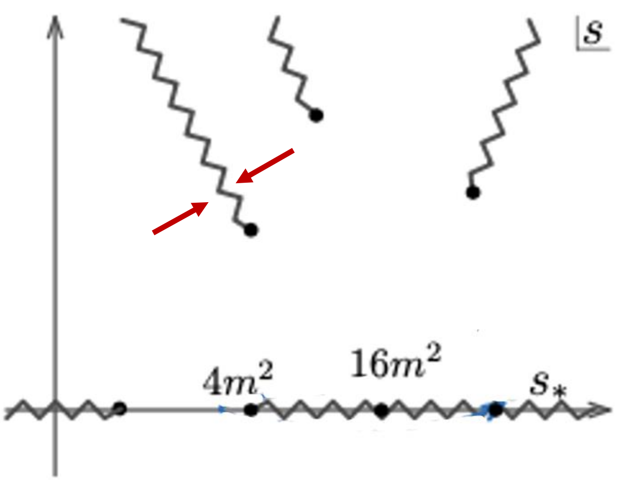
Absorption integrals

Cutkosky: The discontinuity of an integral is given by an **absorption integral** where all the cut lines are replaced by δ functions

$$\mathcal{A}_G^\kappa(p) = \int \prod_{c\in\widehat{C}(G)} \mathrm{d}^d k_c \prod_{e\in E_{\rm int}(G^\kappa)} (-2\pi i)\ \theta_*(q_e^0)\delta(q_e^2 - m_e^2) \prod_{e'\in E(G)\setminus E(G^\kappa)} \frac{1}{q_{e'}^2 - m_{e'}^2 + i\varepsilon}\,.$$



- Cutkosky's formula isolates individual branch points/cuts

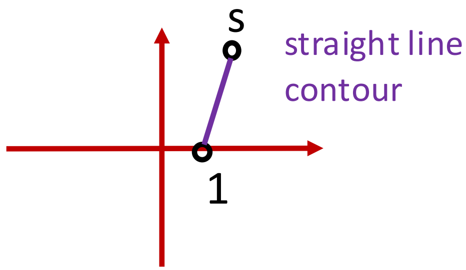


- θ and δ functions make formula ambiguous for complex momenta
- Formula actually only applies for "principal" singularities, which include all physical ones

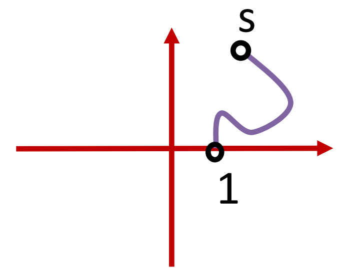
Where does this formula come from?
What is the cleanest way to understand it?

Monodromies

$$\ln s = \int_1^s \frac{dx}{x}\,.$$



$$\ln_\gamma s = \int_\gamma \frac{dx}{x}$$



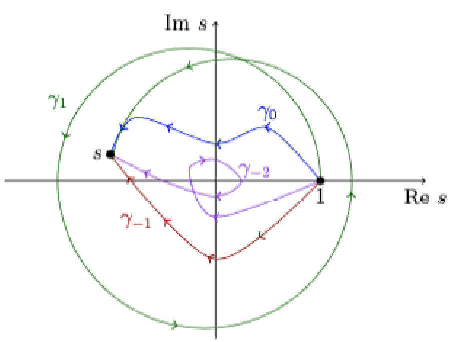
equal to conventional definition of ln s

- contour cannot pass through x=0
- not defined for $s < 0$

generalization to arbitrary contour γ

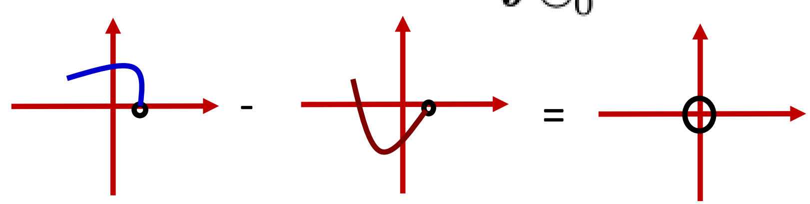
- contour cannot pass through x=0
- s<0 is ok
- branch cut no longer exists
- called the maximal analytic continuation of ln s

- small deformations of contour cannot change the result
- contours classified by winding number around branch point x=0



difference between contours is **monodromy** = disc = im

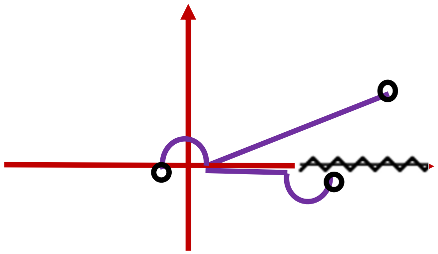
$$\ln_{\gamma_0} s - \ln_{\gamma_{-1}} s = \int_{\circlearrowleft_0} \frac{dx}{x} = 2\pi i$$



Dilogarithm

- Principal branch defined with straight line contours
  - Singularities avoided counterclockwise

$$\mathrm{Li}_2(s) = -\int_0^s \frac{dx}{x}\ln(1-x') = \int_0^s \frac{dx}{x}\int_0^{x'} \frac{dx'}{1-x'}$$



Branch point at s=1

- discontinuity along the branch cut for s>1 computed via monodromy:

$$\mathrm{Disc}\,\mathrm{Li}_2(s) = \int_0^s \frac{dx}{x}\int_{\circlearrowleft_1} \frac{dx'}{1-x'} = 2\pi i \int_0^s \frac{dx}{x} = 2\pi i \ln s$$

- monodromy around s=1
- monodromy around s=0 vanishes

Now branch point at s=0 is visible

- Branch point at s=0 is on second sheet

Singularities encoded transparently with the **symbol**

$$\mathrm{Li}_s(s) = \int d\ln s \int d\ln(1-s) = \int_{\gamma_0} \frac{ds}{1-s} \circ \frac{ds}{s}$$

$$\mathcal{S}(\mathrm{Li}_2) = (1-s) \otimes s$$

first discontinuity sequential discontinuity

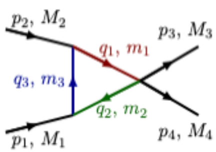
Landau Equations

Associate a Feynman integral to a graph G

e.g.

numerator = 1 for simplicity

$$I_G(p) = \int \prod_{c \in \widehat{C}(G)} \mathrm{d}^d k_c \prod_{e \in E_{\text{int}}(G)} \frac{1}{[q_e(k,p)]^2 - m_e^2 + i\varepsilon}$$

G= 

Go to Feynman parameters

$$I_G(p) = (n_{\text{int}} - 1)! \int_0^\infty \prod_{e \in E_{\text{int}}(G)} \mathrm{d}\alpha_e \int \prod_{c \in \widehat{C}(G)} \mathrm{d}^d k_c \frac{1}{(\ell + i\varepsilon)^{n_{\text{int}}}} \delta\Big(1 - \sum_{e \in E_{\text{int}}(G)} \alpha_e\Big)$$

internal edges

internal edges

fundamental cycles
(independent loop momenta)

$$\ell = \sum_{e \in E_{\text{int}}(G)} \alpha_e (q_e^2 - m_e^2)$$

Where in the space of external momenta p is the graph singular?

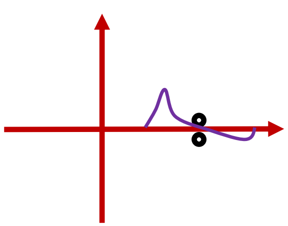
A *necessary* condition for a singularity is that the *integrand* is singular ($\ell = 0$)

$$\int_2^{10} dx \frac{1}{x - 3 + i\varepsilon} = \ln(-7 - i\varepsilon)$$

Not singular

$$\int_2^{10} dx \frac{1}{(x-3)^2 + i\varepsilon} \sim \frac{1}{\sqrt{\varepsilon}} = \infty$$

Singular



integration contour
pinched between poles

Landau Equations

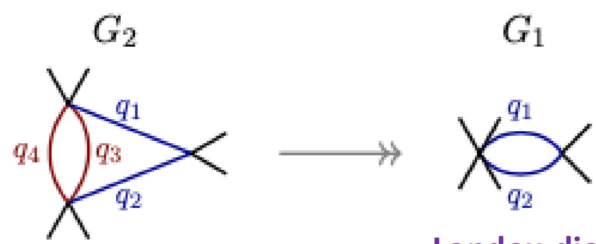
$$I_G(p) = (n_{\text{int}} - 1)! \int_0^\infty \prod_{e \in E_{\text{int}}(G)} \mathrm{d}\alpha_e \int \prod_{c \in \widehat{C}(G)} \mathrm{d}^d k_c \frac{1}{(\ell + i\varepsilon)^{n_{\text{int}}}} \delta\Big(1 - \sum_{e \in E_{\text{int}}(G)} \alpha_e\Big)$$

A necessary condition for a singularity is that the *integrand* is singular ($\ell=0$)

$$\boxed{\ell = \sum_{e \in E_{\text{int}}(G)} \alpha_e (q_e^2 - m_e^2) \ = 0}$$

- every internal line is either on-shell ($q^2=m^2$) or $\alpha=0$ or both

consider only the lines with $\alpha \neq 0$



Landau diagram

q_1, q_2 on-shell. q_3,q_4 irrelevant

Landau Equations

$$I_G(p) = (n_{\rm int}-1)! \int_0^\infty \prod_{e\in E_{\rm int}(G)} {\rm d}\alpha_e \int \prod_{c\in\widehat{C}(G)} {\rm d}^d k_c \frac{1}{(\ell + i\varepsilon)^{n_{\rm int}}} \delta\Big(1 - \sum_{e\in E_{\rm int}(G)} \alpha_e\Big)$$

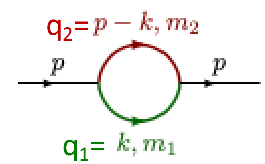
A necessary condition for a singularity is that the *integrand* is singular ($\ell=0$)

$$\boxed{\ell = \sum_{e\in E_{\rm int}(G)} \alpha_e (q_e^2 - m_e^2) = 0}$$

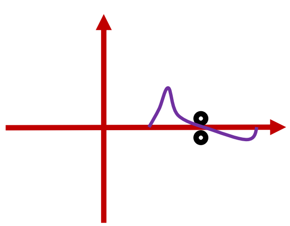
- every internal line is either on-shell ($q^2=m^2$) or $\alpha=0$ or both

A necessary condition for a singularity of the *integral* is that there be double poles

for each loop k_c: $$\sum_{e\in E_{\rm int}(G^\kappa)} \alpha_e \frac{\partial}{\partial k_c}(q_e^2 - m_e^2) = 0\,.$$



Double pole:



integration contour
pinched between poles

- since q_e are linear in k_c

$$\Rightarrow \boxed{\sum_{e\,{\rm in\,loop}} \pm\alpha_e q_e^\mu = 0}$$ Landau loop equations

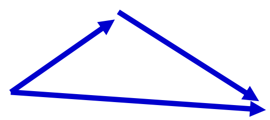
Coleman-Norton interpretation

Landau equations

$$\ell = \sum_{e \in E_{\text{int}}(G)} \alpha_e (q_e^2 - m_e^2) = 0$$

$$\sum_{e \,\text{in}\, \text{loop}} \pm \alpha_e q_e^\mu = 0$$

4-momenta add up to zero after rescaling by α



[Coleman and Norton 1965]

Landau diagram is interpreted as space-time diagram

- momenta are on-shell (classical)
- α_e are the proper times for propagation

More physically: singularities due to classically allowed processes

- similar to optical thoerem

Pham interpretation

Landau equations

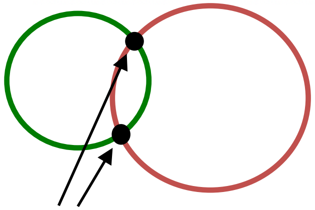
$$\ell = \sum_{e \in E_{\rm int}(G)} \alpha_e (q_e^2 - m_e^2) = 0$$

$$\sum_{e \,\mathrm{in\, loop}} \pm \alpha_e q_e^\mu = 0$$

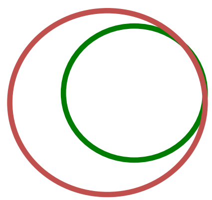
⇒ normal vectors of on-shell constraints $q^2=m^2$ are linearly dependent

on-shell constraints (Euclidean d=2)

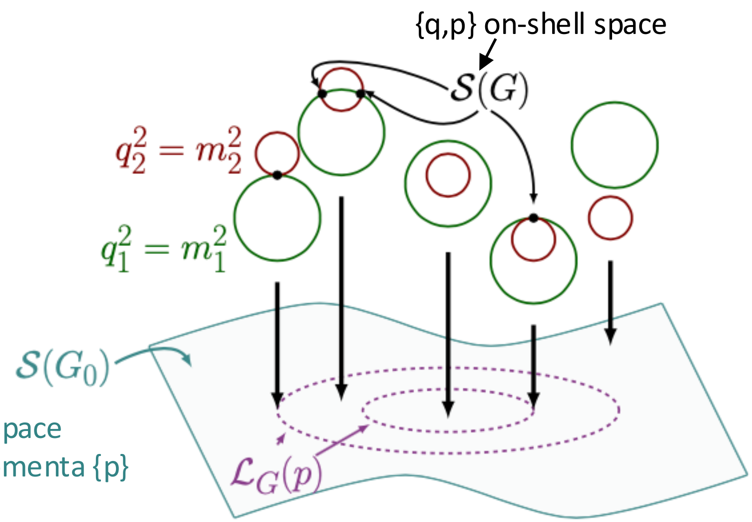
$$q_x^2 + q_y^2 = m_e^2$$



intersection satisfies both on-shell constraints



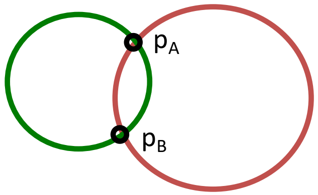
tangent on boundary of space where circles intersect



Landau variety is the boundary of the projection map

Vanishing cycle

Mathematics



consider integration contours in the space

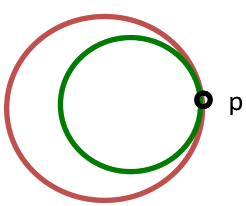
$$\mathcal{E}(G) \setminus \mathcal{S}(G)$$

space of momenta with on-shell locus removed

Homology group is that of a plane with two holes

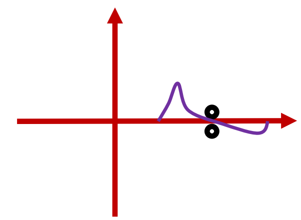
$$H_1(\mathbb{R}^2 \setminus \{p_A \cup p_B\})$$

When cirlces are tangent homology group shrinks



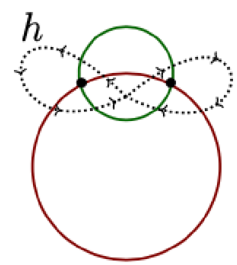
$$H_1(\mathbb{R}^2 \setminus p)$$

Physics



singularity is pinched as ε→0

homology cycle h becomes trivial



Hadamard's "vanishing cycle"

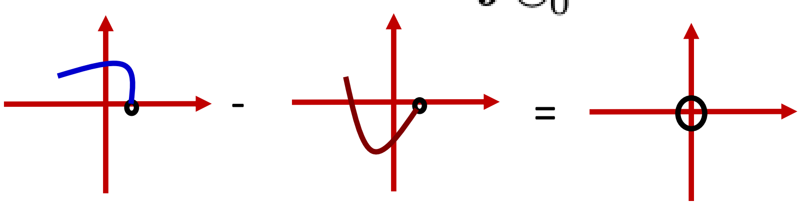
Homology and Homotopy

Integrals are functions of external momenta p

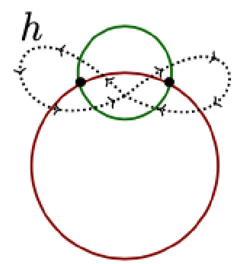
$$I_G(p) = \int \prod_{c\in\widehat{C}(G)} \mathrm{d}^d k_c \prod_{e\in E_{\mathrm{int}}(G)} \frac{1}{[q_e(k,p)]^2 - m_e^2 + i\varepsilon}$$

Homotopy classes of paths in space of external momenta determine discontinuities

$$\ln_{\gamma_0} s - \ln_{\gamma_{-1}} s = \int_{\circlearrowleft_0} \frac{dx}{x} = 2\pi i$$



Homology classes in space of internal momenta determines branch points



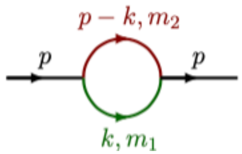
These two concepts are connected through the Picard-Lefshetz theorem

Homology: integration contours γ and γ’ are homologous if γ-γ’ is a boundary of some space
Homotopy: homotopic paths in external momenta can be deformed into each other

1-loop example

Consider again the 1-loop bubble in d=2

$$I_{\bigcirc}(p) =$$



$$= \lim_{\varepsilon \to 0^+} \int \mathrm{d}^2 k \frac{1}{k^2 - m_1^2 + i\varepsilon} \frac{1}{(p-k)^2 - m_2^2 + i\varepsilon},$$

Going to Feynman parameters

$$I_{\bigcirc}(s) = \lim_{\varepsilon \to 0^+} \int_0^1 \mathrm{d}\alpha \frac{-i\pi}{s\alpha(1-\alpha) - m_1^2\alpha - m_2^2(1-\alpha) + i\varepsilon} \quad \text{'}=\ell$$

integrand is singular ($\ell = 0$) at

$$\alpha_{\pm} = \frac{s + m_2^2 - m_1^2 \pm \sqrt{s^2 - 2s(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 + is\varepsilon}}{2s}.$$

on-shell locus

- necessary but not sufficient condition for singularities of integral

demanding also that $\frac{d\ell}{d\alpha} = 0$ gives two solutions

normal threshold $s = (m_1 + m_2)^2 - i\varepsilon, \qquad \alpha_{\pm} = \frac{m_2}{m_2 + m_1} + i\varepsilon \, \mathrm{sgn}(m_2 - m_1),$

pseudothreshold $s = (m_1 - m_2)^2 + i\varepsilon, \qquad \alpha_{\pm} = \frac{m_2}{m_2 - m_1} - i\varepsilon \, \mathrm{sgn}(m_2 - m_1).$

- location of branch points
- solutions to Landau equations

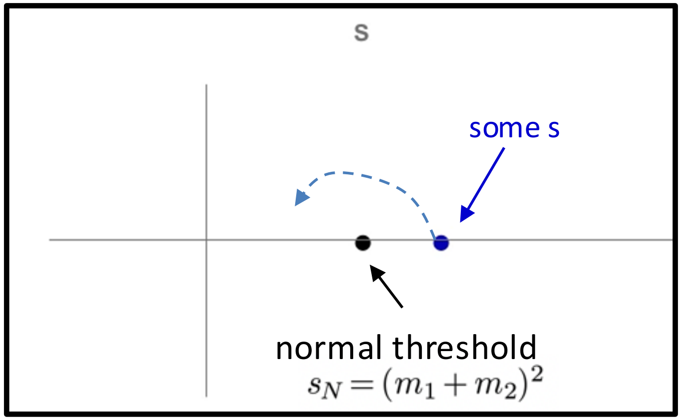
Picard-Lefschetz Theorem

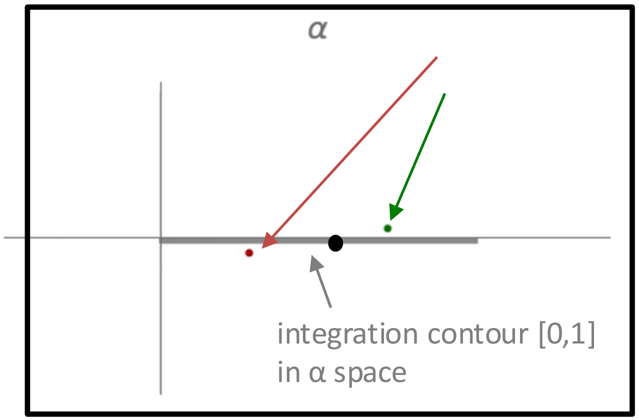
$$I_{\bigcirc}(s) = \lim_{\varepsilon\to 0^+}\int_0^1 \mathrm{d}\alpha \frac{-i\pi}{s\alpha(1-\alpha) - m_1^2\alpha - m_2^2(1-\alpha) + i\varepsilon}$$

$$= \int_0^1 \frac{d\alpha}{[\alpha-\alpha_+(s)][\alpha-\alpha_-(s)]}$$

on-shell locus: $\alpha = \alpha_\pm$

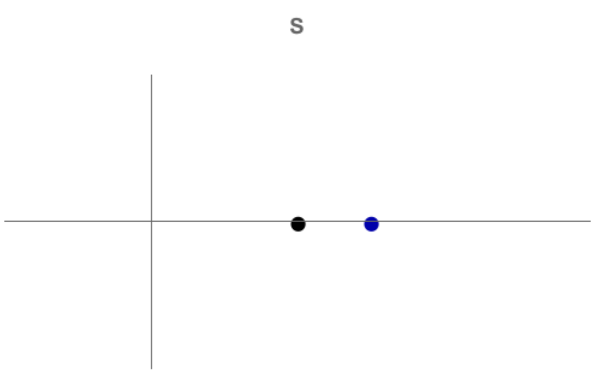
$$\alpha_\pm = \frac{s + m_2^2 - m_1^2 \pm \sqrt{s^2 - 2s(m_1^2+m_2^2) + (m_1^2-m_2^2)^2 + is\varepsilon}}{2s}.$$

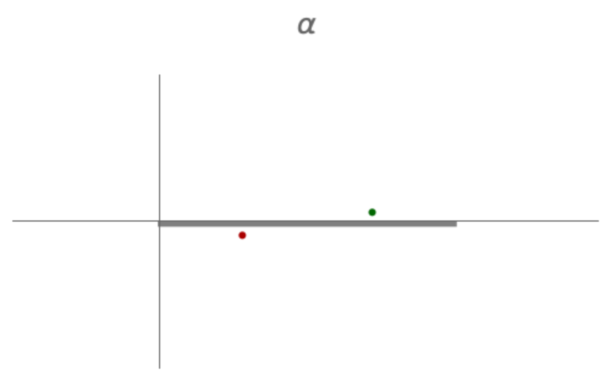




What happens as we take a monodromy of s around s_N?

- Poles $\alpha_\pm$ move around too
- Contour must move out of the way to avoid poles

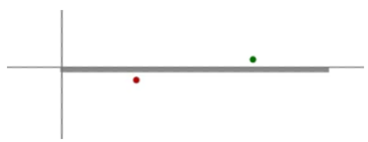




Picard-Lefschetz Theorem

Discontinuity
= difference between I(s) before and after analytic continuation
= monodromy of s around s_N: $\left(\mathbb{1} - \mathscr{M}_{s=(m_1+m_2)^2}\right) I_{\circ}(s_0)$

initial integration contour



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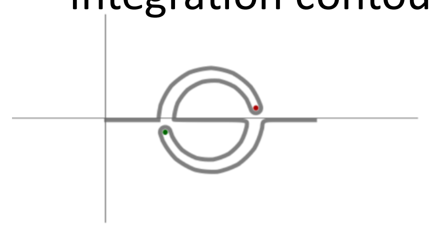
final integration contour



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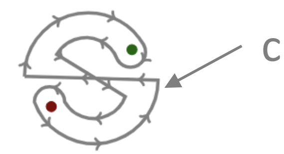
difference integration contour



deform difference contour

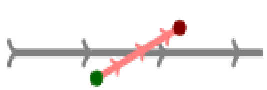
Picard-Lefschetz Theorem:

$$\left(\mathbb{1} - \mathscr{M}_{s=(m_1+m_2)^2}\right) I_{\circ}(s_0) = \langle e, h\rangle \int_c \mathrm{d}I$$

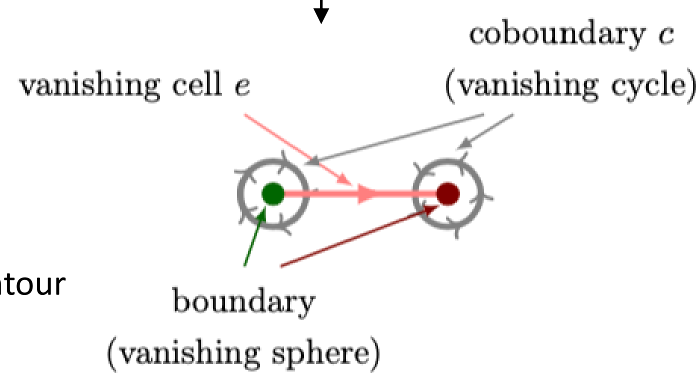
monodromy
in exeternal momtena

integral over
the coboundary

kronecker index = intersection number between integration contour and vanishing cell



<e,h>=1



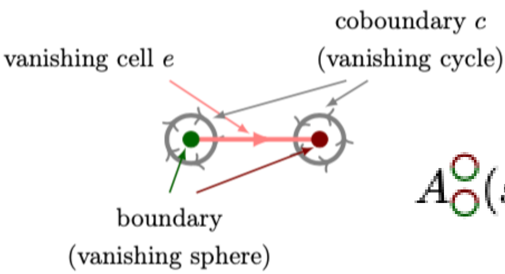
- cycle pinches (vanishes) at $s=s_N$

Picard-Lefschetz Theorem

$$A = (\mathbb{1} - \mathscr{M}_{s=s^*}) \int_h \mathrm{d}I = N_0 \int_c \mathrm{d}I \,,$$

Why do this?

1. treats integral and discontinuity of integral on the same footing
2. formula is fully analytic: no δ functions of θ functions



- Integral over vanishing cycle can be done with Cauchy's thm

$$A_{\bigcirc}^{\bigcirc}(s_0) \equiv \left(\mathbb{1} - \mathscr{M}_{s=(m_1+m_2)^2}\right) I_{\bigcirc}(s_0) = \int_{\odot\ \odot} \mathrm{d}I$$

$$= -\frac{2\pi^2}{s_0}\left(\mathrm{res}_{\alpha=\alpha_+} \frac{\mathrm{d}\alpha}{(\alpha-\alpha_+)(\alpha-\alpha_-)} - \mathrm{res}_{\alpha=\alpha_-} \frac{\mathrm{d}\alpha}{(\alpha-\alpha_+)(\alpha-\alpha_-)}\right)$$

$$\mathrm{res}_{\alpha=\alpha_0} f(\alpha) d\alpha = 2\pi i\, f(\alpha_0) = 2\pi i\, d\alpha \delta(\alpha - \alpha_0)$$

For >1d integrals, need Leray multivariate residue calculus

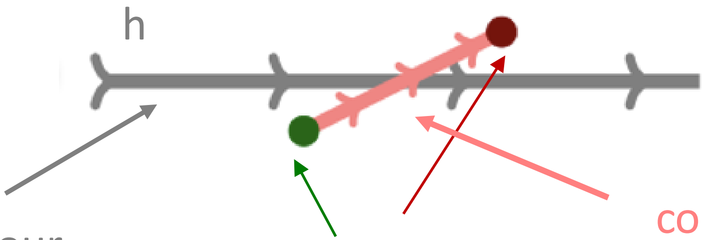
- Leads to Cutkosky's formula

$$\int_{\delta_1 \delta_2 \sigma} \omega = (2\pi i)^2 \int_\sigma \mathrm{res}_{S_2}\, \mathrm{res}_{S_1}\, \omega .$$

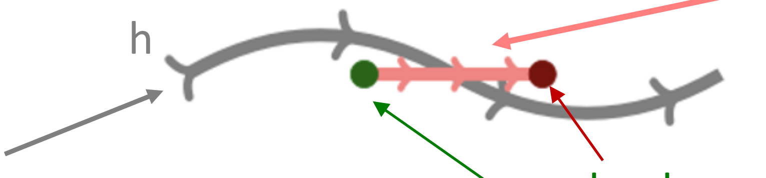
Leray coboundary operator

Imaginary contours

Physicists like to keep +iε in the interand and integration contour d^4k real



We can deform



Integrand is now real

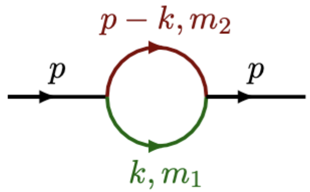
$$I_{\bigcirc}(p) = \int_h \frac{\mathrm{d}^2 k}{[k^2 - m_1^2][(p-k)^2 - m_2^2]}$$

No more iε

Deformation doesn't even have to be small

Momentum space

- More useful, and physical, to work in momentum space instead of α space

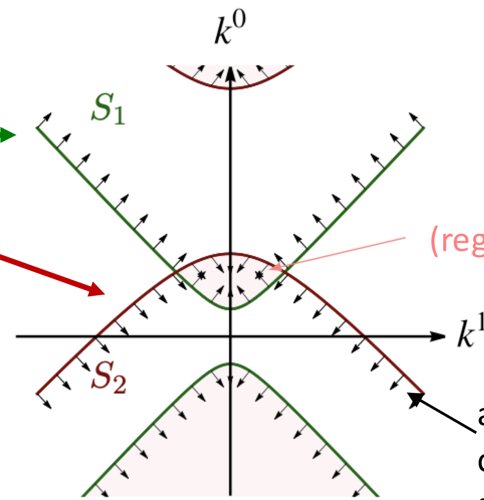
$$I_{\bigcirc}(p) =$$ 

$$= \int_h \frac{\mathrm{d}^2 k}{[k^2 - m_1^2][(p-k)^2 - m_2^2]}$$

In d=2 on-shell spaces are hyberbolas

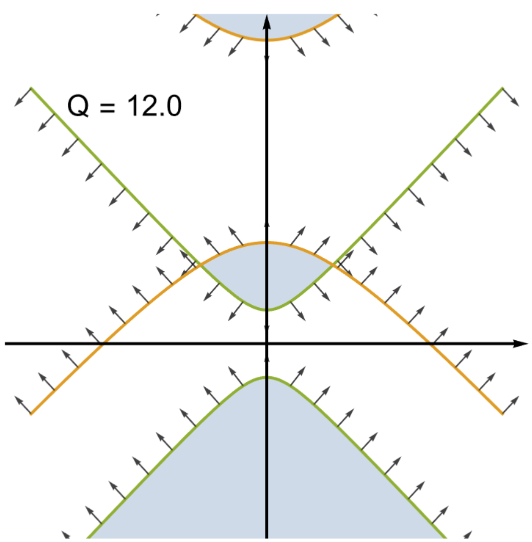
$$s_1(p,k) = (k^0)^2 - (k^1)^2 - m_1^2\,,$$

$$s_2(p,k) = (Q - k^0)^2 - (k^1)^2 - m_2^2\,.$$

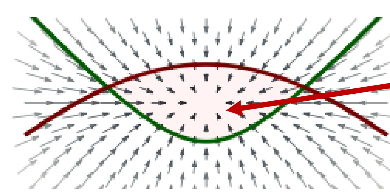


vanishing cell
(region with $s_1>0$ & $s_2>0$)

arrows indicate
complex part from +iε
- integration contour is real plane

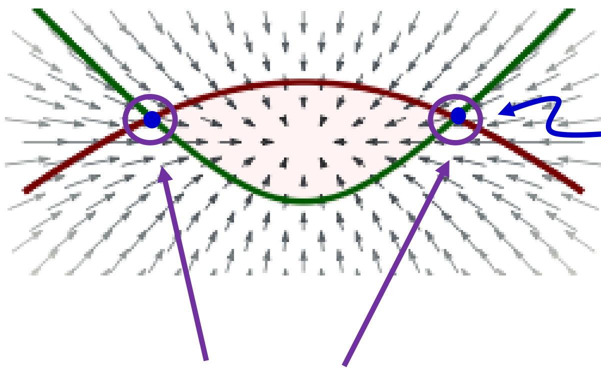


go to complex integration contour



must be real at some point
- must pinch singular surface

Vanishing cell/cycle/sphere



1. Define vanishing cell **e** by $s_1>0$ & $s_2>0$
2. Define vanishing sphere as $\partial_1 \partial_2$ **e**

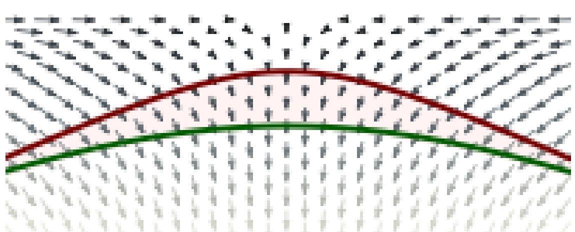
same as surface where all lines are on-shell

$\partial_1 \partial_2 e = S_1 \cap S_2$

3. Define the **vanishing cycle** as the coboundary $\delta_1 \delta_2$ of the vanishing sphere

$$\left(1-\mathscr{M}_{s=(m_1+m_2)^2}\right) I_{\bigcirc}(p) = -\langle e,h\rangle \int_{\delta_1\delta_2\partial_2\partial_1 e} \frac{\mathrm{d}k_0\wedge \mathrm{d}k_1}{[k^2-m_1^2][(p-k)^2-m_2^2]}$$

discontunity = integral over vanishing cycle



for pesudothreshold

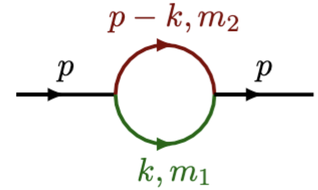
- integration region does not intersect vanishing cell
- discontinuity vanishes

Bubble in d=3

on-shell locus

$$s_1 = k_0^2 - \vec{k}^2 - m_1^2 = 0$$

$$s_2 = (Q - k_0)^2 - \vec{k}^2 - m_2^2 = 0$$

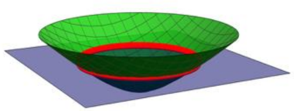
$$I_{\circ}(p) = $$ 

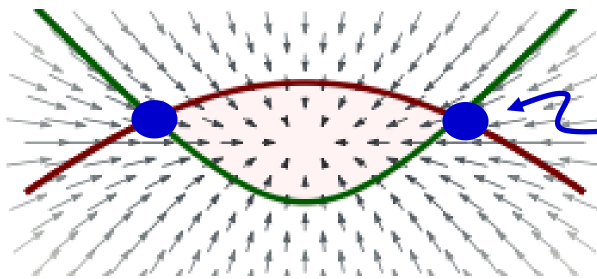
$$= \frac{\sqrt{\pi}}{\sqrt{s}} \log\left(\frac{m_1 + m_2 + \sqrt{s}}{m_1 + m_2 - \sqrt{s}}\right)$$

magnitude of $\vec{k}$ fixed: $|\vec{k}|^2 = \dfrac{(Q + m_1 + m_2)(Q - m_1 - m_2)(Q - m_1 + m_2)(Q + m_1 - m_2)}{4Q^2}$

at fixed Q

- 2 points for d=2
- circle for d=3



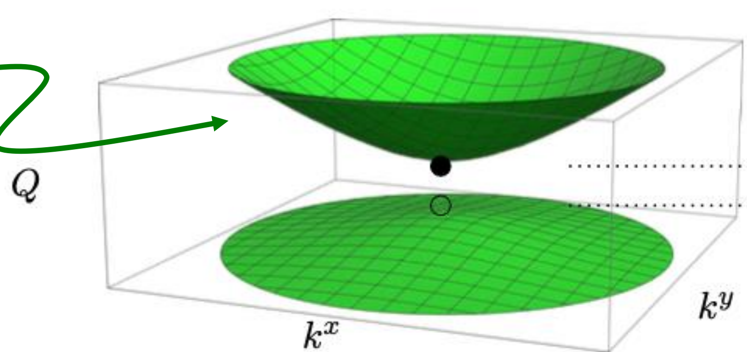


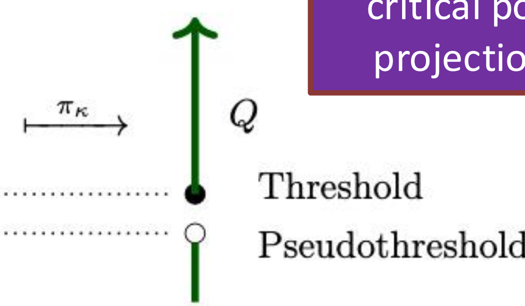
vanishing sphere (d=2)

as a function of Q (external momentum): paraboloid

branch points are critical points of projection map

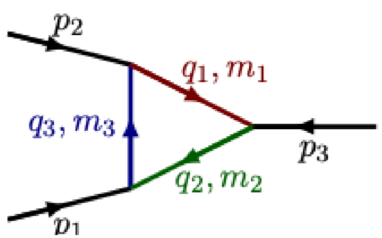
vanishing sphere (on-shell space)





Triangle in d=3

$I_{\triangleright}(p) =$



$$= \int_h \mathrm{d}^3k \frac{1}{s_1(p,k)s_2(p,k)s_3(p,k)}$$

$$s_i(p,k) = q_i^2(p,k) - m_i^2$$

$$= \frac{\pi^2}{4\sqrt{D}}\left[\log\left(-\frac{y_{12}+y_{23}\,y_{13}+i\sqrt{D}}{y_{12}+y_{23}\,y_{13}-i\sqrt{D}}\right)+\log\left(-\frac{y_{23}+y_{13}\,y_{12}+i\sqrt{D}}{y_{23}+y_{13}\,y_{12}-i\sqrt{D}}\right)+\log\left(-\frac{y_{13}+y_{12}\,y_{23}+i\sqrt{D}}{y_{13}+y_{12}\,y_{23}-i\sqrt{D}}\right)+\pi i\right]$$

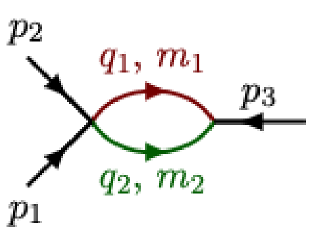
$$y_{ij} = \frac{(p_i+p_j)^2 - m_i^2 - m_j^2}{2m_i m_j}$$

$$D = 1 - y_{12}^2 - y_{23}^2 - y_{13}^2 - 2y_{12}\,y_{23}\,y_{13}$$

Landau variety $\ell = \sum_{e\in E_{\mathrm{int}}(G)} \alpha_e(q_e^2 - m_e^2)$ =0 has 4 branches

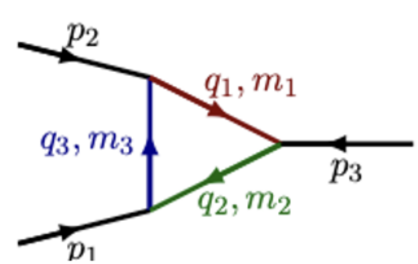
3 bubble singularities

- One of the α_e=0
- corresponds to y_{ij}=1



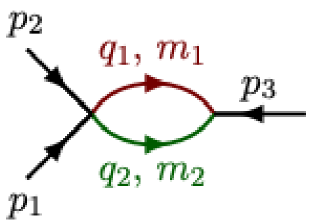
1 triangle singularity

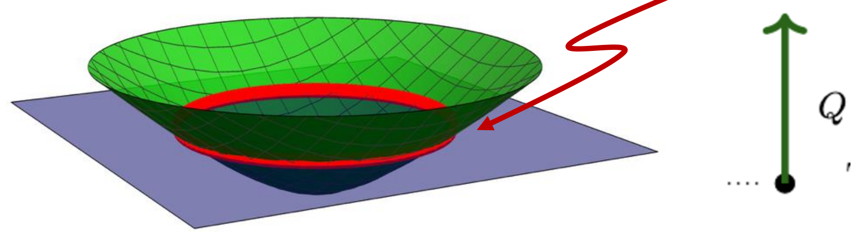
- All α_e >0
- Corresponds to D=0



Bubble singularity of the triangle

The on-shell space (vanishing sphere)
for the bubble at fixed external momenta Q =$(p_3)^2$= is a circle





Absorption integral for the bubble singularity (y_{12}=1) of the triangle
i.e. monodromy around y_{12}=1

vanishing cycle

$$A_{\triangleright}^{\ominus}(p) = (1-\mathscr{M}_{y_{12}=1})\, I_{\triangleright} = -\langle e_{12}, h\rangle \int_{\delta_1\delta_2\partial_2\partial_1 e_{12}} \omega\,.$$

vanishing sphere

Leray residue formula

$$= -(2\pi i)^2 \langle e_{12}, h\rangle \int_{\partial_2\partial_1 e_{12}} \operatorname{res}_2 \operatorname{res}_1 \omega$$

$$= -(2\pi i)^2 \langle e_{12}, h\rangle \int_{\partial_2\partial_1 e_{12}} \frac{\mathrm{d}^3 k}{(\mathrm{d}s_1 \wedge \mathrm{d}s_2) s_3}.$$

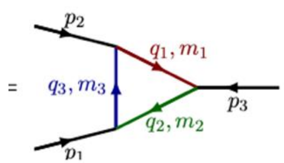
can write as an integral over the vanishing sphere

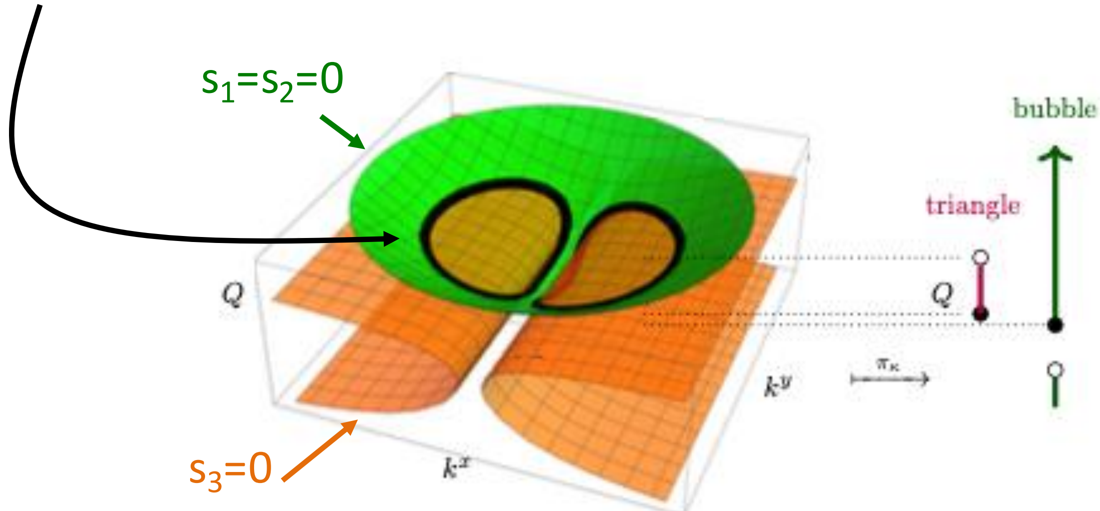
Sequentual discontinuity

Now we want to take a discontinuity of $A_{\triangleright}^{\bigcirc}$ around the triangle singularity (D=0)

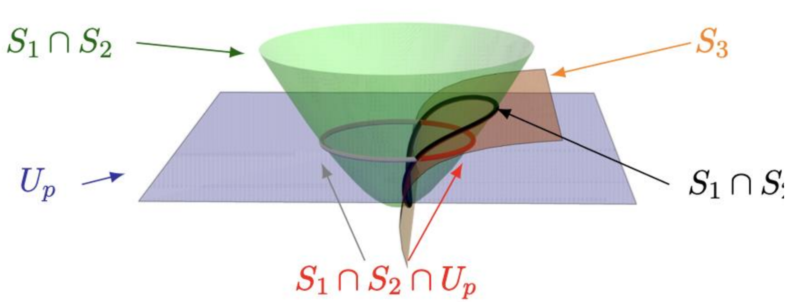
$$(1-\mathscr{M}_{D=0})\, A_{\triangleright}^{\bigcirc}(p)$$

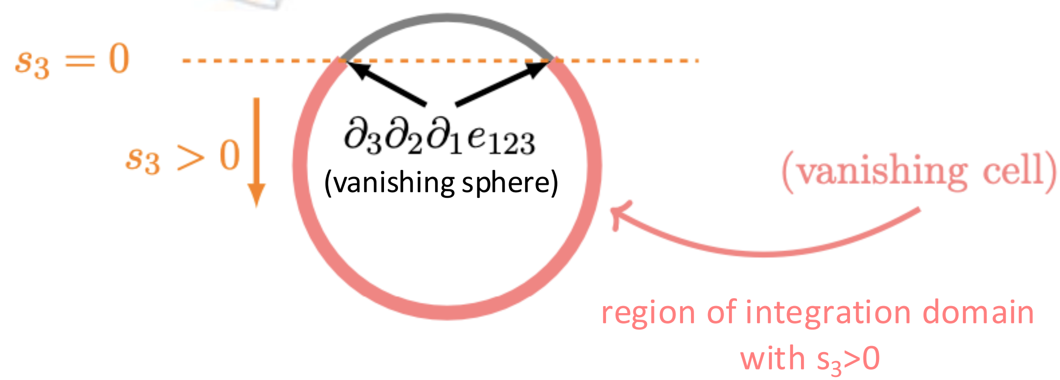
The **on-shell space** now has all 3 propagators on shell: $s_1=s_2=s_3=0$





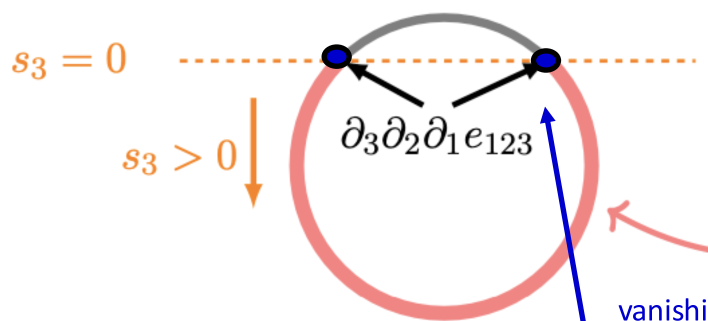
Fix Q (intersect with a plane)





integration domain for bubble integral

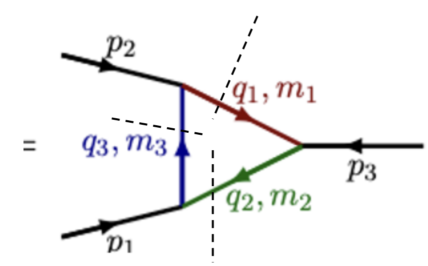
Sequential discontinuity



region of integration domain with s_3>0

(vanishing cell)

vanishing sphere



3 δ-functions for 3 cut lines

$$(1-\mathscr{M}_{D=0})\, A_{\triangleright}^{\circ}(p) = (2\pi i)^2 \langle e_{12|3}, \partial_2\partial_1 e_{12}\rangle \langle e_{12}, h\rangle \int_{\delta_3\partial_3\partial_2\partial_1 e_{123}} \frac{\mathrm{d}^3 k}{(\mathrm{d}s_1 \wedge \mathrm{d}s_2) s_3}$$

$$= (2\pi i)^3 \int_{\partial_1\partial_2\partial_3 e_{123}} \frac{\mathrm{d}^3 k}{\mathrm{d}s_1 \wedge \mathrm{d}s_2 \wedge \mathrm{d}s_3} = \frac{2\pi^3 i}{\sqrt{D}}$$

We get the same thing as if we just took a single triangle discontinuity:

$$(1-\mathscr{M}_{D=0}) I_{\triangleright} = -\langle e_{123}, h\rangle \int_{\delta_1\delta_2\delta_3\partial_3\partial_2\partial_1 e_{123}} \frac{\mathrm{d}^3 k}{s_1 s_2 s_3}$$

$$= (2\pi i)^3 \int_{\partial_3\partial_2\partial_1 e_{123}} \frac{\mathrm{d}^3 k}{\mathrm{d}s_1 \wedge \mathrm{d}s_2 \wedge \mathrm{d}s_3} = \frac{2\pi^3 i}{\sqrt{D}}$$

hierarchcial Pham relation

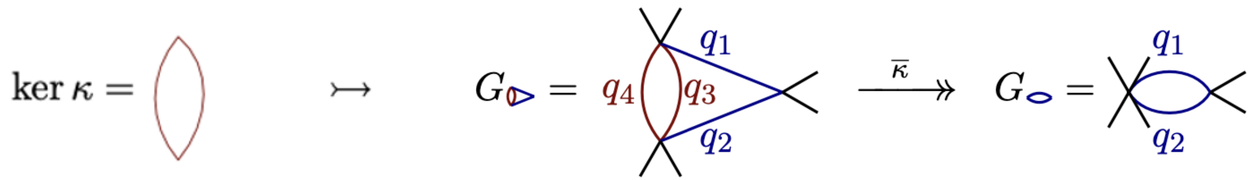
$$\left(\mathbb{1}-\mathscr{M}_{D=0}\right)\left(\mathbb{1}-\mathscr{M}_{y_{12}=1}\right) I_{\triangleright} = \left(\mathbb{1}-\mathscr{M}_{D=0}\right) I_{\triangleright} .$$

Contractions

A useful language for studying singularities of integrals is with **graph contractions**

removing legs and connecting vertices

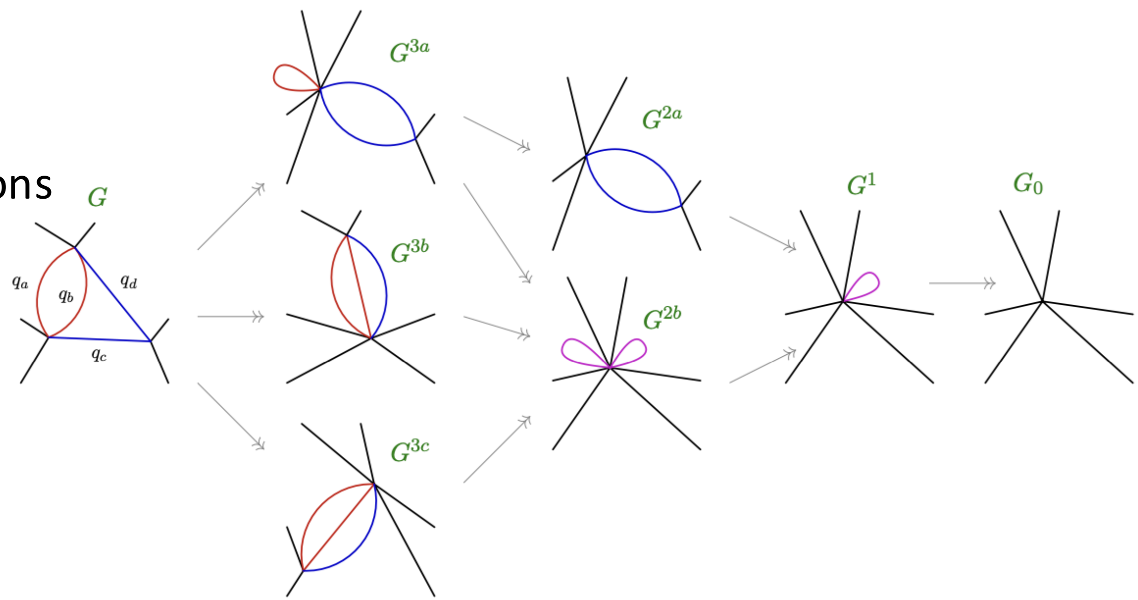
short
exact sequence



kernel of contraction

image of contraction

All Landau diagrams
come from contractions
of original graph

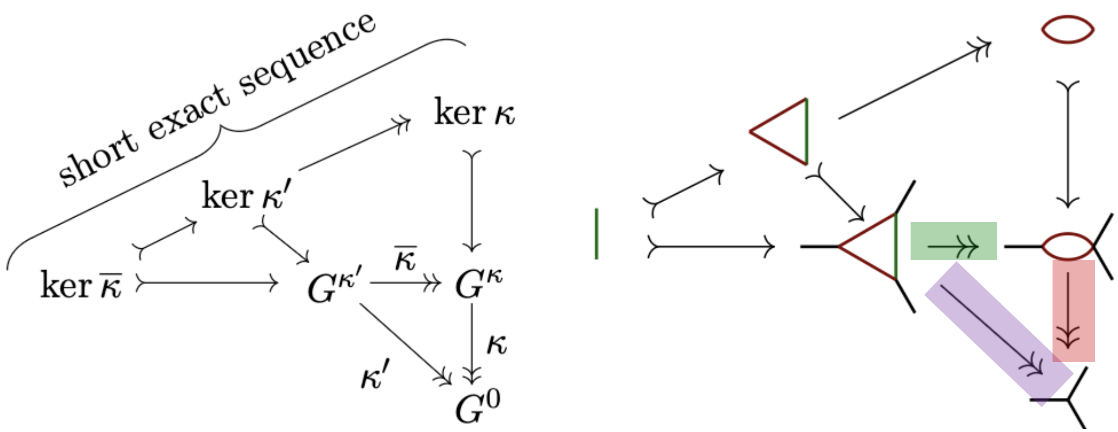


Hierarchical principle

Theorem 2 (Pham). *For a series of contractions* $G \twoheadrightarrow G^{\kappa'} \twoheadrightarrow \cdots \twoheadrightarrow G^{\kappa} \twoheadrightarrow G_0$ *the relation*

$$\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa'}}\right)\cdots\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa}}\right)I_G(p)=\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa'}}\right)I_G(p) \tag{6.2}$$

holds when $\mathcal{P}_{\kappa}\cdots\mathcal{P}_{\kappa'}$ *correspond to principal Pham loci, and* p *is in the physical region.*



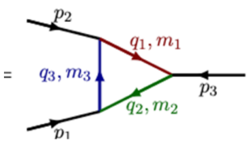
e.g. $(\mathbb{1}-\mathscr{M}_{D=0})(\mathbb{1}-\mathscr{M}_{y_{12}=1})I_{\triangleright} = (\mathbb{1}-\mathscr{M}_{D=0})I_{\triangleright}$.

- “Principal” is a technical mathematical requirement about stable topological type
- Pham corrected subtlety in previous formulation of the “hierarchical principle”

[Landsdoff et al. 1966]

Tangential intersections

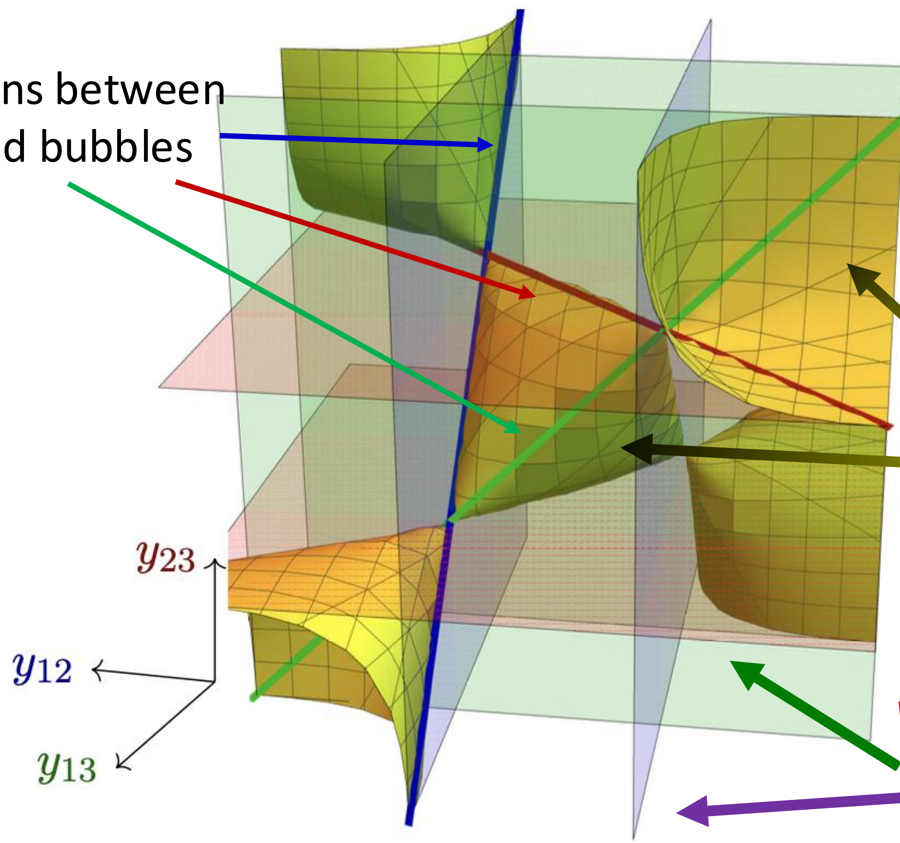
For the d=3 bubble, look at the Landau variety in the space of external kinematics



$$y_{ij} = \frac{(p_i + p_j)^2 - m_i^2 - m_j^2}{2 m_i m_j}$$

$$D = 1 - y_{12}^2 - y_{23}^2 - y_{13}^2 - 2 y_{12}\, y_{23}\, y_{13}$$

lines are intersections between triangle and bubbles

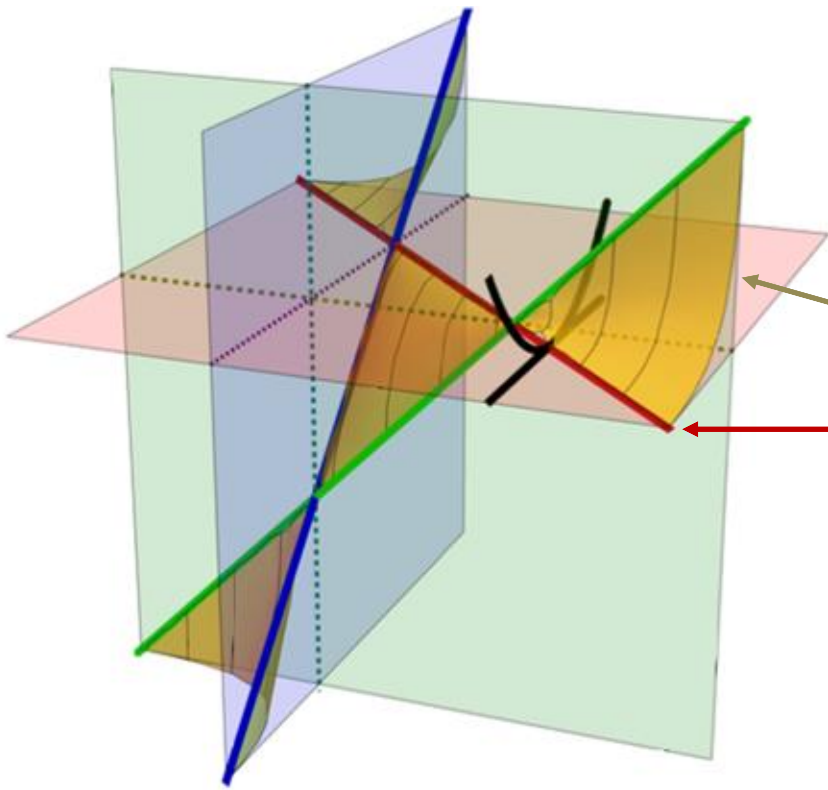


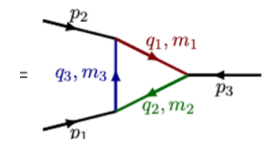
pillow region and cones are D=0 (triangle)

planes are $y_{ij}=\pm 1$ (bubbles)

Tangential intersections

For the d=3 bubble, look at the Landau variety in the space of external kinematics for α>0





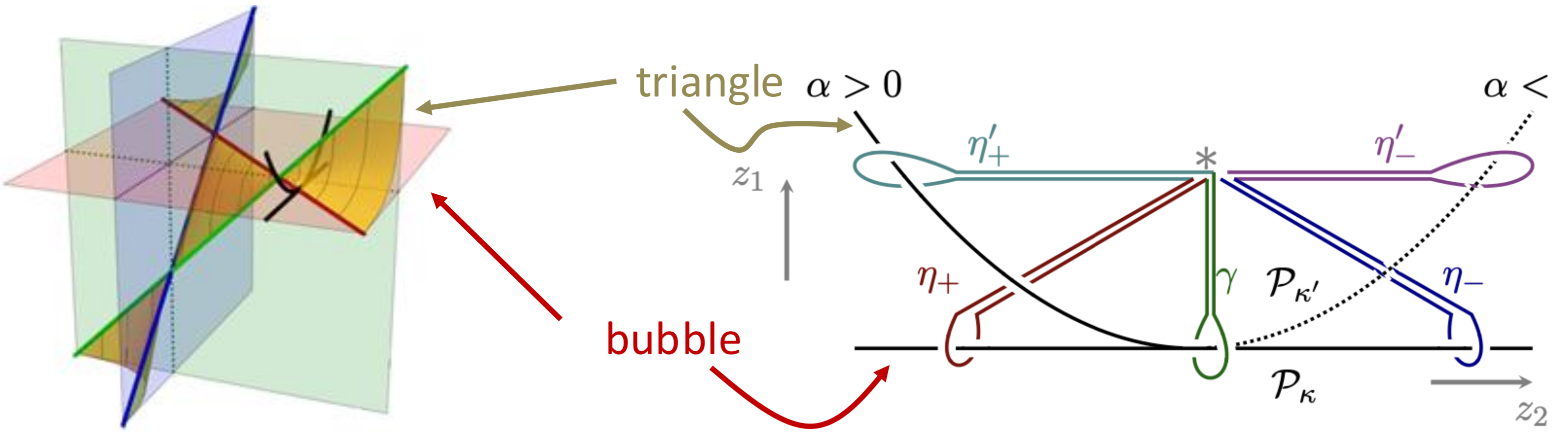
$$y_{ij} = \frac{(p_i + p_j)^2 - m_i^2 - m_j^2}{2 m_i m_j}$$

$$D = 1 - y_{12}^2 - y_{23}^2 - y_{13}^2 - 2 y_{12}\, y_{23}\, y_{13}$$

bubble and triangle
intersect tangentially

regions shown have α>0
(singularities are in the physical region)

Tangential intersections



We can consider monodromies around bubble and triangle

We want to take $(\mathbb{1} - \mathscr{M}_{D=0})(\mathbb{1} - \mathscr{M}_{y_{12}=1}) I_{\triangleright}$

$$\mathscr{M}_{D=0} = \mathscr{M}_{\eta'_+}, \qquad \mathscr{M}_{y_{12}=1} = \mathscr{M}_{\eta_+}$$

no singularity for α<0
adding monodromy does nothing

$$\mathscr{M}_{\eta'_-} I_G(p) = I_G(p).$$

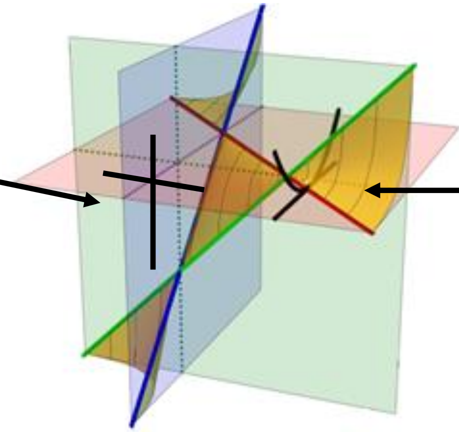
We can show that

$$\eta'_+ \circ \eta_+ = \eta_+ \circ \eta'_- \implies \mathscr{M}_{\eta'_+} \circ \mathscr{M}_{\eta_+} I_G(p) = \mathscr{M}_{\eta_+} \circ \mathscr{M}_{\eta'_-} I_G(p) = \mathscr{M}_{\eta_+} I_G(p)\,.$$

$$\implies \boxed{\left(\mathbb{1} - \mathscr{M}_{\eta'_+}\right)\left(\mathbb{1} - \mathscr{M}_{\eta_+}\right) I_G(p) = \left(\mathbb{1} - \mathscr{M}_{\eta_+}\right) I_G(p)}$$

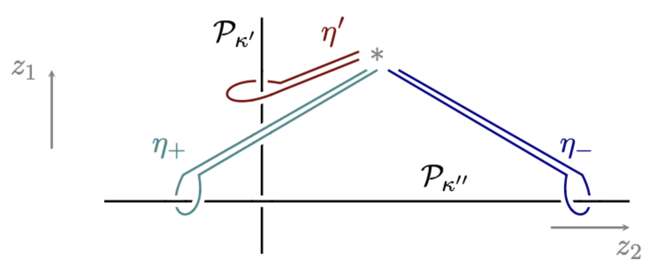
Transversal intersections

bubbles intersect each other transversely



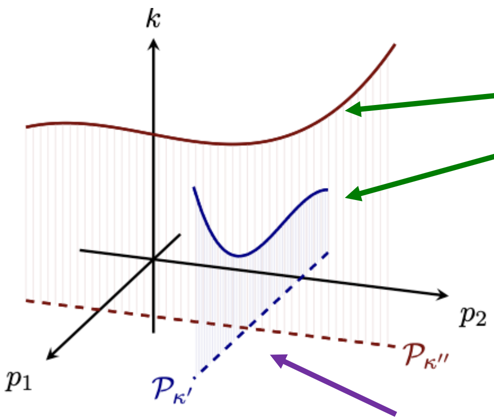
traingle and bubble intersect tangentially

For transversal intersections, monodromies commute



Thm (Pham): $$\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa'}}\right)\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa''}}\right)I_G(p)=\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa''}}\right)\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa'}}\right)I_G(p)$$

Transversal intersections



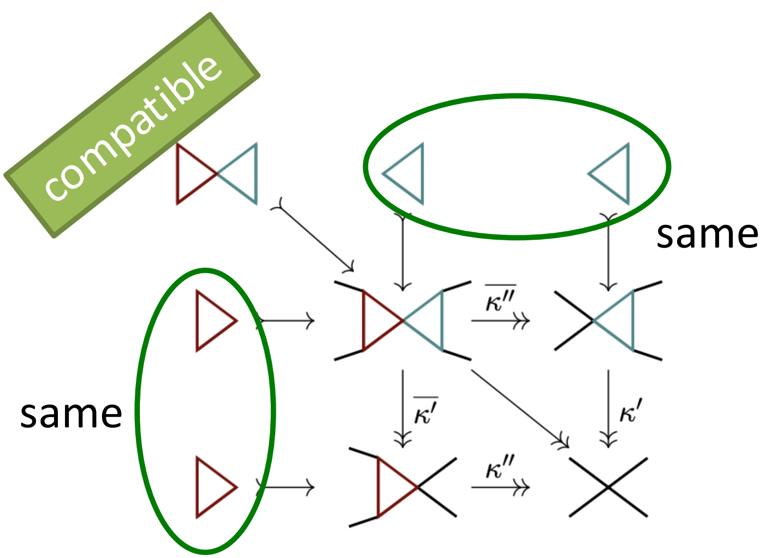
on-shell surfaces may not intersect in *internal momenta*

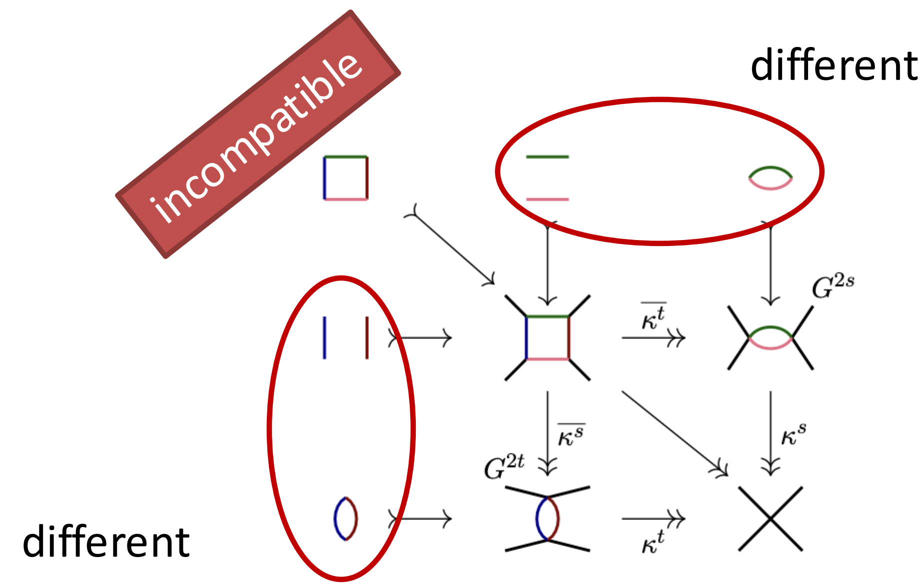
- vanishing cell from first monodromy doesn't intersect integration contour of second
- sequential monodromy vanishes

$$\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa'}}\right)\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa''}}\right)I_G(p)=0$$

Singular surfaces intersect transversaly in external momenta

- Condition for internal intersecton is
  - Landau equations for internal momenta can be solved simultaneously
  - Kernels of contractions are compatible (Pham)

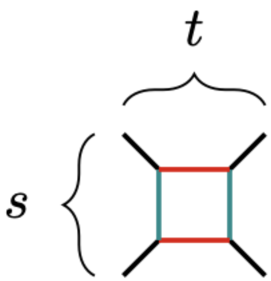




Steinmann relations

No sequential discontinuities in partially overlapping channels in the physical region

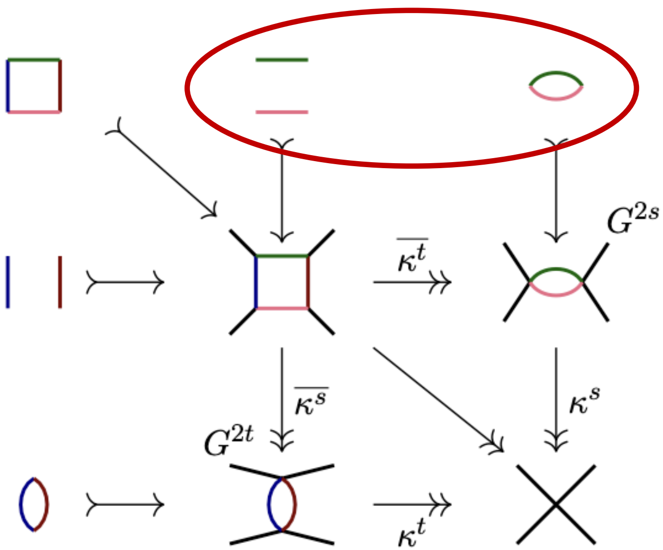
[Steinmann 1960, in German]



cannot have a term like

$$\log(s - 4m^2)\log(t - 4m^2)$$

Follows from the Pham diagram analysis

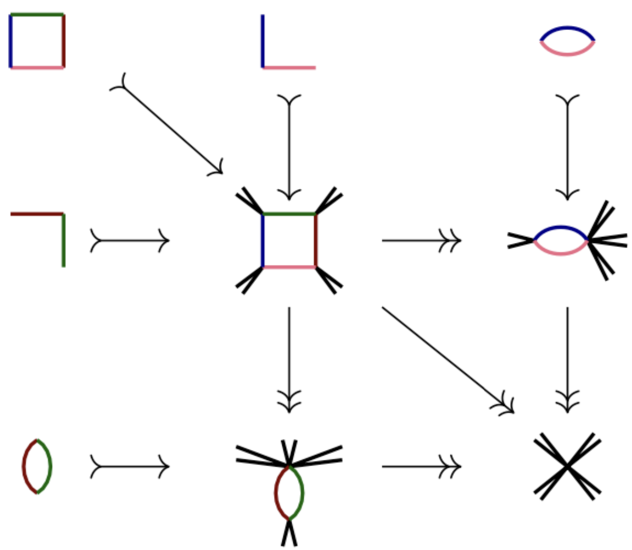


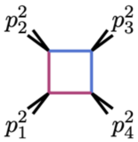
kernels are incompatible

$$\left(\mathbb{1} - \mathscr{M}\right)\left(\mathbb{1} - \mathscr{M}\right) I_G(p) = 0$$

Extended Steinmann relations

Compatible kernel condition more general



Box diagram 

cannot have terms like

$$\log(p_1^2 - 4m^2)\log(p_3^2 - 4m^2)$$

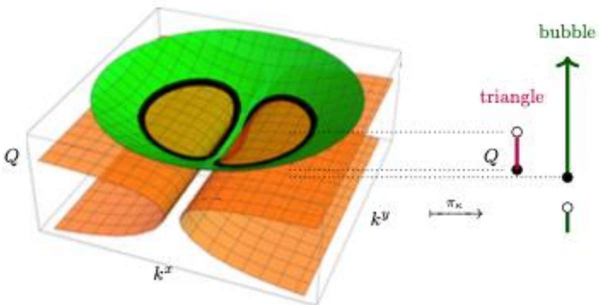
- channels are not partially overlapping

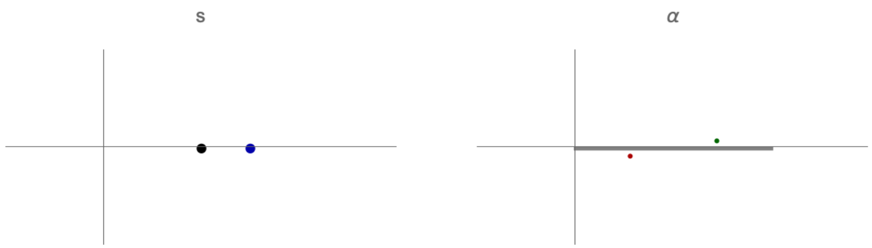
Summary

Geometric analyis is a powerful way to understand singularities of scattering amplitudes

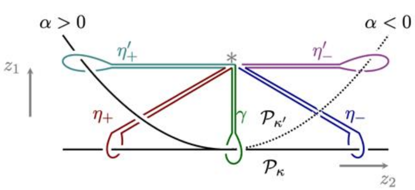
1. Branch points are critical points of projection map

2. Picard-Lefshetz and Leray coboundary theory connect homotopy of paths in external momenta to homology of integration contours





3. Geometric picture lets us prove general relations about sequential discontinuities

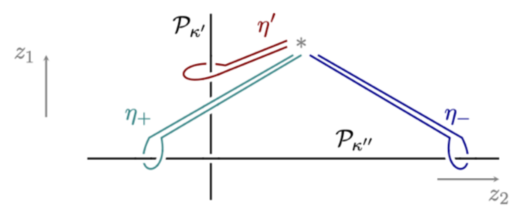


hierarchical case (tangential)

$$\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa'}}\right)\cdots\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa}}\right)I_G(p)=\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa'}}\right)I_G(p)$$

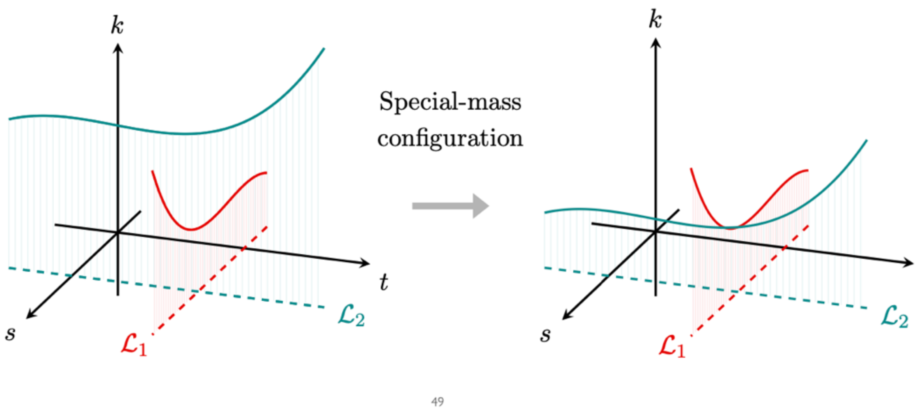
non-hierarchical case (transversal)

$$\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa'}}\right)\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa''}}\right)I_G(p)=\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa''}}\right)\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa'}}\right)I_G(p)$$



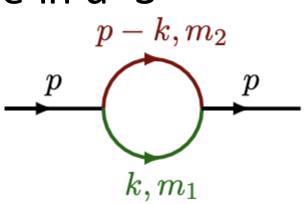
Next steps

- Weaken assumptions
  - We assumed all masses were generic
  - Zero masses, or equal mass can make singularities overlap



- Study second-type (non-Landau) singularities

  e.g. bubble in d=3

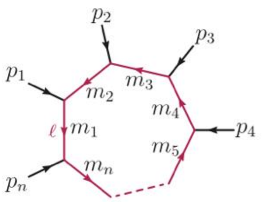
$I_{\bigcirc}(p) =$ 

$= \dfrac{\sqrt{\pi}}{\sqrt{s}} \log\left(\dfrac{m_1 + m_2 + \sqrt{s}}{m_1 + m_2 - \sqrt{s}}\right)$

  - Not on physical sheet
  - Still relevant to analytic structure of scattering amplitudes

Next steps

- Study more examples
  - All mass n-gon in n-dimensions (like bubble in 2d, triangle in 3d)



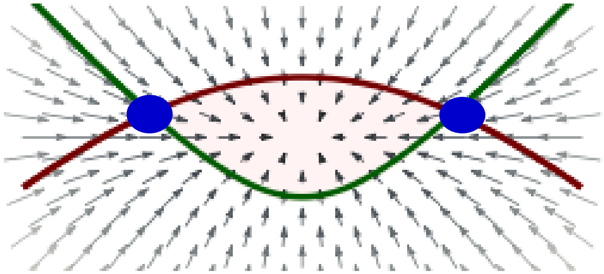
$$\mathcal{S}\left(I_n^{\text{one loop}}\right) \propto \frac{1}{\sqrt{\det y}} \sum \omega^{\varnothing}_{\{i_1,i_2\}} \otimes \omega^{\{i_1,i_2\}}_{\{i_1,i_2,i_3,i_4\}} \otimes \cdots \otimes \omega^{\{i_1,\ldots,i_{n-2}\}}_{\{1,\ldots,n\}}$$

$$\text{cut}_J\, I_n^{\text{one loop}}(y) = \frac{(2\pi i)^{|J|}}{\sqrt{\det y}} \sqrt{\det y'}\; I^{\text{one loop}}_{n-|J|}(y')$$

- Connect back to the finite S matrix
  - Can overlapping singularities tell us about factorization?
  - Does preserving Pham relations in the massless limit lead to a natural scheme for remainder functions?
  - What can be said non-perturbatively?

Older

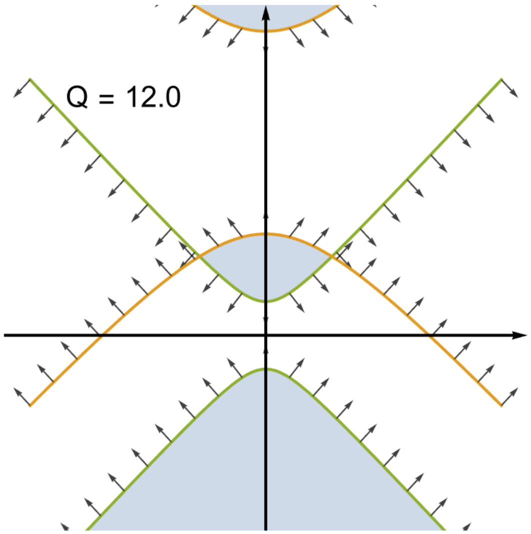
In d=2, vasnishing sphere (on-shell locus) is two points

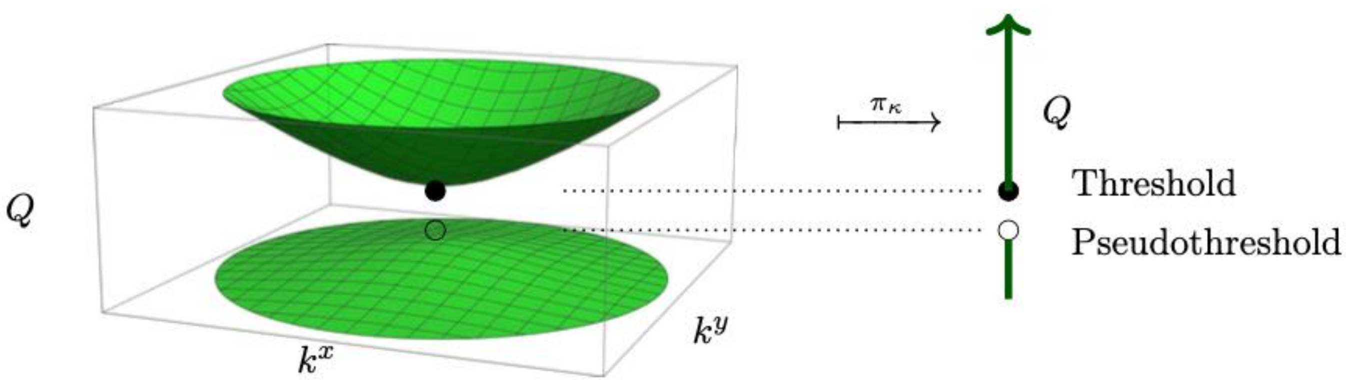


$$s_1 = k_0^2 - \vec{k}^2 - m_1^2 = 0$$

$$s_2 = (Q - k_0)^2 - \vec{k}^2 - m_2^2$$

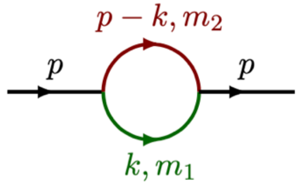
as Q varies points approach and retreat





Example

Consider the simplest 1-loop diagram: the bubble in d=2

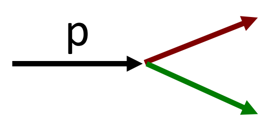
$I_{\bigcirc}(p) =$ 

$$= \lim_{\varepsilon\to 0^+} \int \mathrm{d}^2 k \frac{1}{k^2 - m_1^2 + i\varepsilon} \frac{1}{(p-k)^2 - m_2^2 + i\varepsilon}$$

$$= \frac{-2\pi}{\sqrt{-[s-(m_1-m_2)^2][s-(m_1+m_2)^2]}} \log\left(\frac{\sqrt{(m_1+m_2)^2 - s} - i\sqrt{s-(m_1-m_2)^2}}{\sqrt{(m_1+m_2)^2 - s} + i\sqrt{s-(m_1-m_2)^2}}\right)$$

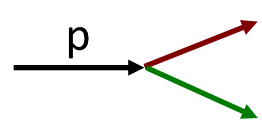
Even this diagram is remarkably rich, as we will see.

- At has a **normal threshold** branch cut starting at $s = s_N = (m_1 + m_2)^2$
  - For $s > s_N$ the on-shell process $p \to p_1 + p_2$ is allowed for physical on-shell momenta



  - Tree-level process tells you about singularities of loop amplitudes
  - e.g., through optical theorem

Im 

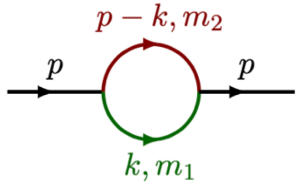
$= \int d\Pi \,\Big|$ 

$\Big|^2$

Example

Consider the simplest 1-loop diagram: the bubble in d=2

$$I_{\bigcirc}(p) =$$



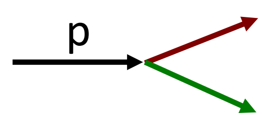
$$= \lim_{\varepsilon \to 0^+} \int \mathrm{d}^2 k \frac{1}{k^2 - m_1^2 + i\varepsilon} \frac{1}{(p-k)^2 - m_2^2 + i\varepsilon}$$

$$= \frac{-2\pi}{\sqrt{-[s-(m_1-m_2)^2][s-(m_1+m_2)^2]}} \log\left(\frac{\sqrt{(m_1+m_2)^2 - s} - i\sqrt{s-(m_1-m_2)^2}}{\sqrt{(m_1+m_2)^2 - s} + i\sqrt{s-(m_1-m_2)^2}}\right)$$

Even this diagram is remarkably rich, as we will see.

It has a branch cut starting at $s = s_N = (m_1 + m_2)^2$

- This is a **normal threshold**
  - For $s > s_N$ the on-shell process $p \to p_1 + p_2$ is allowed for physical on-shell momenta
  - Near normal threshold



$$I_{\bigcirc}(p) : \longrightarrow -\frac{2\pi}{\sqrt{-4m_1 m_2 (s - s_N)}} \ln(-1)$$

- It has a **pseudothreshold** branch cut starting at $s = s_P = (m_1 - m_2)^2$
  - Cannot be reached with physical momenta (real s>0,)
  - Near pseudothreshold

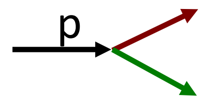
$$I_{\bigcirc}(p) : \longrightarrow -\frac{2\pi}{\sqrt{4m_1 m_2 (s - s_P)}} \ln(1) = 0$$

BACKUP

Absorption integrals

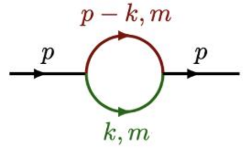
Optical theorem

Im 

= $\int d\Pi \left| \right.$ 

$\left. \right|^2$

All imaginary parts come from iε in propagators $\quad \mathrm{Im}\,\frac{1}{p^2-m^2+i\varepsilon}=2\pi\delta(p^2+m^2)$

$I_{\bigcirc}(p) =$ 

$= \int d^2k \, \frac{1}{k^2-m^2+i\varepsilon} \frac{1}{(p-k)^2-m^2+i\varepsilon}$

Absorption integral

replace propagators with δ functions

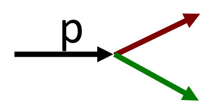
$$A_{\bigcirc}^{\bigcirc}(s_0) = \mathrm{Disc}\; I_{\bigcirc}(p) = 2\,\mathrm{Im}\;\; I_{\bigcirc}(p) = \int d^2k\delta(k^2-m^2)\theta(k_0)[\delta(p-k)^2-m^2]\theta(p_0-k_0)$$

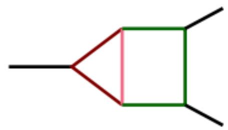
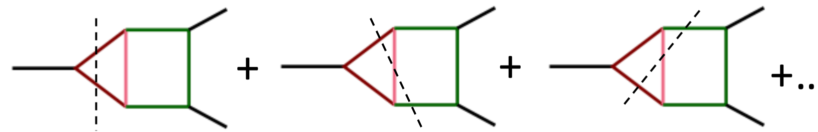
Cutkosky: The discontinuity of an integral is given by an absorption integral where all the cut lines are replaced by δ functions

$$\mathcal{A}_G^{\kappa}(p) = \int \prod_{c\in\widehat{C}(G)} \mathrm{d}^d k_c \prod_{e\in E_{\mathrm{int}}(G^\kappa)} (-2\pi i)\;\theta_*(q_e^0)\delta(q_e^2-m_e^2) \prod_{e'\in E(G)\setminus E(G^\kappa)} \frac{1}{q_{e'}^2-m_{e'}^2+i\varepsilon}\,.$$

Im and Disc

Optical theorem

Im  $= \int d\Pi \, \Big|$  $\Big|^2$

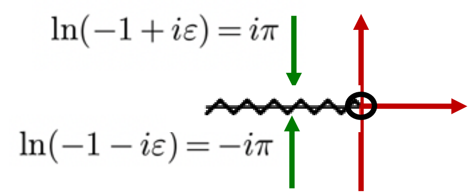
Im  = sum of all cuts 

- Imaginary part is a very coarse tool: cannot isolate individual branch points

Consider conventional definition of ln(z), e.g. in Mathematica

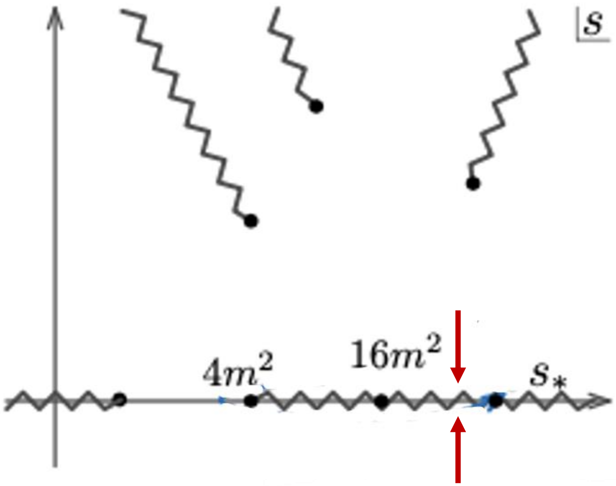
1. Imaginary part defined on negative real axis $\mathrm{Im}\ln(-z) = i\pi$
2. Has a branch point at z=0 and a branch cut for z<0
   - ln(z) is discontinuous across branch cut

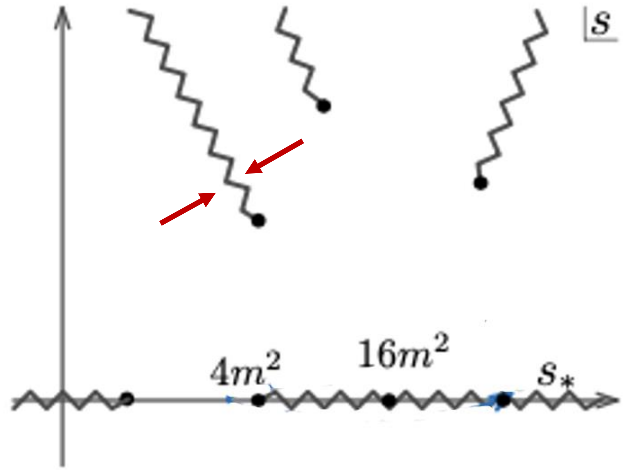
$$\mathrm{Disc}_z \ln z = \ln(z + i\varepsilon) - \ln(z - i\varepsilon) = 2\pi i\,\theta(z)$$



   - Discontinuitiy is twice the imaginary part for ln(z)

Challenges with Im





Imaginary part gives the total discontinuity

- Cannot distinguish overlapping branch cuts

- To understand full analytic structure need to isolate each branch point/cut

$$\mathrm{Im}\,\frac{1}{p^2-m^2+i\varepsilon}=2\pi\delta(p^2+m^2)$$

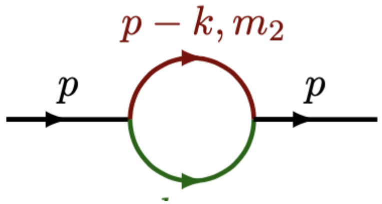
- Imaginary part is real
  - Cannot find sequential discontinnuties by taking imaginary part again

- Absorption integral formula has non-analytic components

Example

Consider the simplest 1-loop diagram: the bubble in d=2

$$I_{\bigcirc}(p) =$$



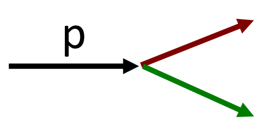
$$= \lim_{\varepsilon \to 0^+} \int \mathrm{d}^2 k \frac{1}{k^2 - m_1^2 + i\varepsilon} \frac{1}{(p-k)^2 - m_2^2 + i\varepsilon}$$

$$= \frac{-2\pi}{\sqrt{-[s-(m_1-m_2)^2][s-(m_1+m_2)^2]}} \log\left(\frac{\sqrt{(m_1+m_2)^2 - s} - i\sqrt{s-(m_1-m_2)^2}}{\sqrt{(m_1+m_2)^2 - s} + i\sqrt{s-(m_1-m_2)^2}} \right)$$

Even this diagram is remarkably rich, as we will see.

It has a branch cut starting at $s = s_N = (m_1 + m_2)^2$

- This is a **normal threshold**
  - For s > s_N the on-shell process p → p_1 +p_2 is allowed for physical on-shell momenta

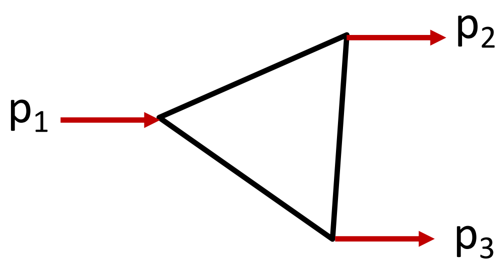


It has another branch cut starting at $s = s_N = (m_1 + m_2)^2$

- This is a **pseudo threshold**
  - Cannot be reached with physical momenta (real s>0,)

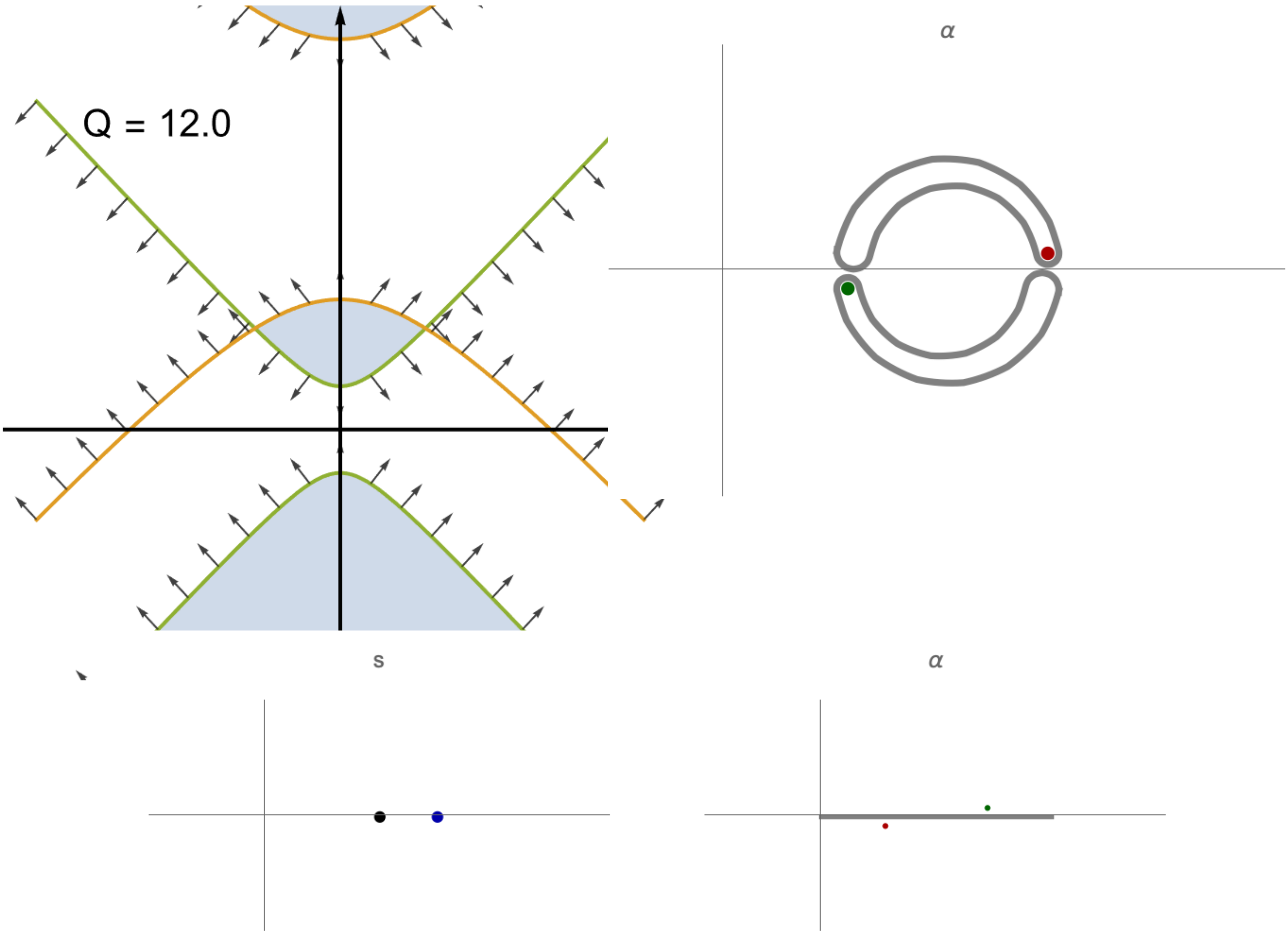
Example

Consider the 3-point diagram at 1-loop in a theory with massless internal lines:



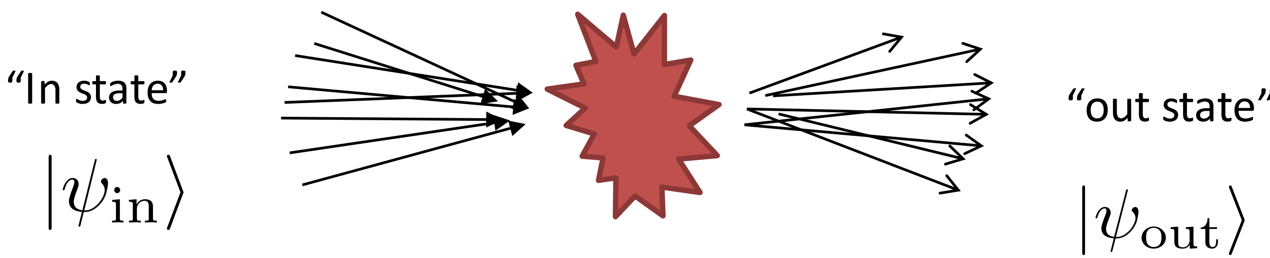
$$= \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\ln(z\bar{z})\ln\left(\frac{1-z}{1-\bar{z}}\right)$$

with $z\bar{z} = p_2^2/p_1^2, \;\; (1-z)(1-\bar{z}) = p_3^2/p_1^2$


Q = 12.0
α
s
α

The S Matrix

The S matrix describes the scattering of particles



- S matrix been studied both perturbatively and non-perturbatively
- **Does it exist?**
  - Hard to prove
  - The usual definition only of S only works for theories with a mass gap

- **Is it unique?**
  - Strong constraints: unitarity, analyticity, causality, cluster-decomposition, etc.
  - The S matrix program of the 1950s-1960s studied this question

- **What constraints does it satisfy?**
  - Useful both pertubatively and non-perturbatively

Analyticity revisited

- The S matrix program from 1960s was never completed
  - Progress was slow
  - Quantum Field Theory was shown capable of explaining strong interactions

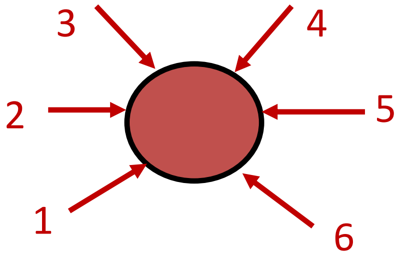
- Recent progress in perurbation theory has renewed interested in analytic structure
  - More "data" – explicit calculations
  - **Mathematics** of functions appearing in amplitudes (cluster algebras, etc.)
    - Very efficient ways to write down amplitudes,
  - Success in the perturbative S-matrix bootstrap
    - collinear limits, Regge limits, conformal invariance, **Steinmann relations**
    - N=4 SYM 6 point amplitude bootstrapped to 7 loops [Caron-Huot et al 1903.10890]

Steinman relations are constraints on sequntial discontinuities [Steinmann 1960]

possible term: $\ln(p_1+p_2)^2 \ln(p_3+p_4)^2$

not allowed (at any order): $\ln(p_1+p_2+p_3)^2 \ln(p_2+p_3+p_4)^2$

Why?

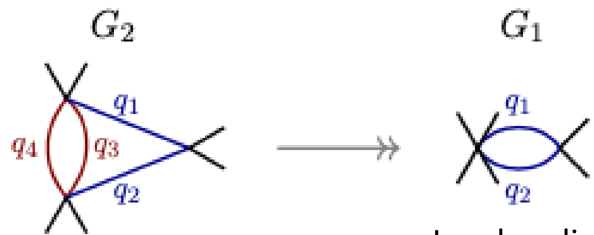


Landau diagrams

consider only the lines with $\alpha \neq 0$

1) Integrand is singular:

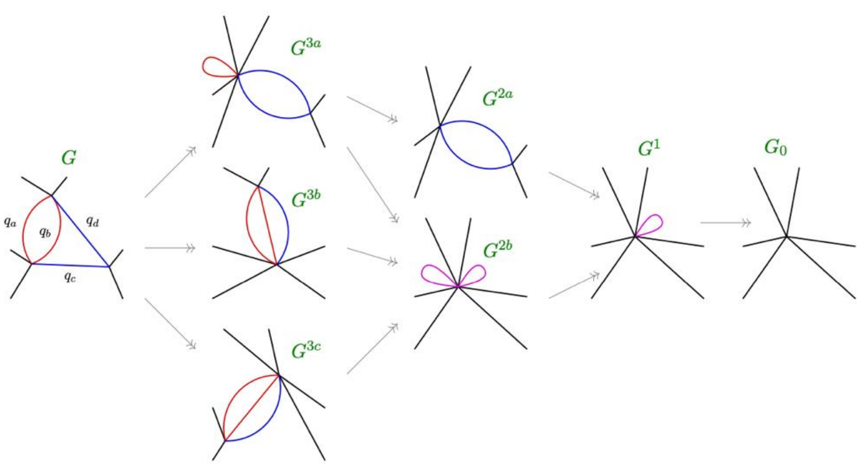
$$\ell = \sum_{e \in E_{\text{int}}(G)} \alpha_e (q_e^2 - m_e^2) = 0$$



Landau diagram

q_1, q_2 on-shell. q_3,q_4 irrelevant

8 Landau diagrams for the ice-cream cone graph



- These are all *possible* branch points (necessary condition only)
- Some diagrams may not be branch points (not a sufficient condition)