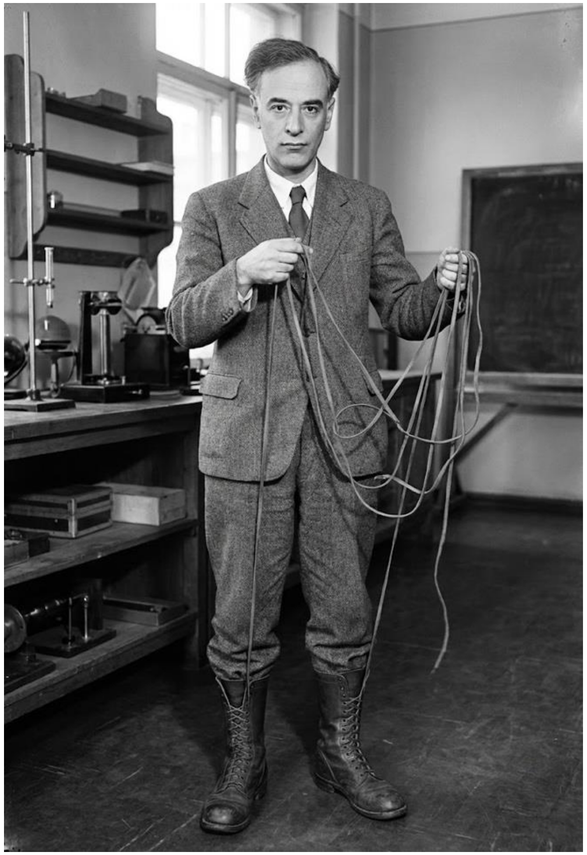
# The Semi-Analytic Landau Boostrap



## FROM ANALYTICITY TO PHENOMENOLOGY

Princeton, NJ, January 12, 2026

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Based on

- H. Hannesdottir, A. McLeod, MDS, C. Vergu:

"Applications of the Landau Bootstrap" 2410.02424

"Constraints on sequential discontinuities from the geometry of on-shell spaces" 2211.07633

- O. Barrera, A. Dersy, R. Husain, MDS and X. Zhang:

"Analytic Regression of Feynman Integrals from High-Precision Numerical Sampling", 2507.17815

# The S-matrix boostrap

**Is the S matrix completely fixed by physical constraints?**

- 1960s: best hope for strong interactions
- 1970s: Quantum Field Theory explained strong interactions

➥S matrix program on hold for 40 years

## Three Modern Bootstraps

### 1. Non-pertubative bootstrap

- e.g. pion 2->2 scattering at low energy
- Similar to 1960's bootstrap
  - Impose unitarity, analyticity, crossing
- Use modern methods

(Hamiltonian truncation, machine learning, etc)

### 2. Perturbative amplitude bootstrap

- constrain full amplitude (sum of all diagrams)
- Impose symmetries and physical limits
  - dual conformal invariance, Regge limit
- 6-point in N=4 SYM at 7 loops computed this way

[Caron-Huot et al 1903.10890]

today's talk

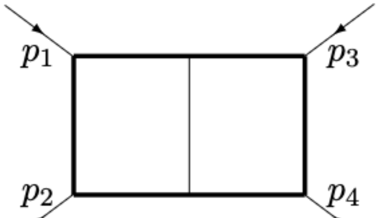
### 3. Feynman diagram bootstrap

- Can we compute Feynman diagrams without actually doing the integrals?
- Many examples to work with
  - Uses elements of 1 and 2

# Feynman-diagram bootstrap

Q: Can we bootstrap Feynman integrals?

Rules: Don't integrate!



$I_5(u,v)$

$$I_5(u,v) = \int \frac{d^Dk_1}{(2\pi)^D} \int \frac{d^Dk_2}{(2\pi)^D} \frac{1}{[k_1^2+m^2]\,[k_2^2+m^2]\,[(k_1-p_1)^2+m^2]\,[(k_2-p_3)^2+m^2]\,[(k_1-k_2)^2]}$$

$$\begin{aligned}
g_{10} =& G_{-1,0}H_{-1,-1} + G_{-1,\frac{z-1}{z+1}}H_{-1,-1} - G_{-1,-\frac{z+1}{z-1}}H_{-1,-1} + 2G_{0,-1}H_{-1,-1} + G_{0,0}H_{-1,-1}\\
&-2G_{0,\frac{1}{z}}H_{-1,-1} - G_{0,\frac{z-1}{z+1}}H_{-1,-1} - G_{0,-\frac{z+1}{z-1}}H_{-1,-1} - G_{1,0}H_{-1,-1} - G_{1,\frac{z-1}{z+1}}H_{-1,-1}\\
&+G_{1,-\frac{z+1}{z-1}}H_{-1,-1} + G_{-1,0}H_{-1,0} - G_{-1,\frac{z-1}{z+1}}H_{-1,0} - G_{-1,-\frac{z+1}{z-1}}H_{-1,0} + G_{0,\frac{z-1}{z+1}}H_{-1,0}\\
&-G_{0,-\frac{z+1}{z-1}}H_{-1,0} - G_{1,0}H_{-1,0} + G_{1,\frac{z-1}{z+1}}H_{-1,0} + G_{1,-\frac{z+1}{z-1}}H_{-1,0} + G_{-1,0}H_{-1,1} + G_{-1,\frac{z-1}{z+1}}H_{-1,1}\\
&-G_{-1,-\frac{z+1}{z-1}}H_{-1,1} + 2G_{0,-1}H_{-1,1} + G_{0,0}H_{-1,1} - 2G_{0,\frac{1}{z}}H_{-1,1} - G_{0,\frac{z-1}{z+1}}H_{-1,1} - G_{0,-\frac{z+1}{z-1}}H_{-1,1}\\
&-G_{1,0}H_{-1,1} - G_{1,\frac{z-1}{z+1}}H_{-1,1} + G_{1,-\frac{z+1}{z-1}}H_{-1,1} - G_{-1,0}H_{0,-1} + G_{-1,\frac{z-1}{z+1}}H_{0,-1} + G_{-1,-\frac{z+1}{z-1}}H_{0,-1}\\
&-G_{0,\frac{z-1}{z+1}}H_{0,-1} + G_{0,-\frac{z+1}{z-1}}H_{0,-1} + G_{1,0}H_{0,-1} - G_{1,\frac{z-1}{z+1}}H_{0,-1} - G_{1,-\frac{z+1}{z-1}}H_{0,-1} + G_{-1,\frac{z-1}{z+1}}H_{0,0}\\
&-G_{-1,-\frac{z+1}{z-1}}H_{0,0} + G_{0,-1}H_{0,0} + G_{0,1}H_{0,0} - G_{0,\frac{z-1}{z+1}}H_{0,0} - G_{0,-\frac{z+1}{z-1}}H_{0,0} - G_{1,\frac{z-1}{z+1}}H_{0,0}\\
&+G_{1,-\frac{z+1}{z-1}}H_{0,0} - G_{-1,0}H_{0,1} + G_{-1,\frac{z-1}{z+1}}H_{0,1} + G_{-1,-\frac{z+1}{z-1}}H_{0,1} - G_{0,\frac{z-1}{z+1}}H_{0,1} + G_{0,-\frac{z+1}{z-1}}H_{0,1}\\
&+G_{1,0}H_{0,1} - G_{1,\frac{z-1}{z+1}}H_{0,1} - G_{1,-\frac{z+1}{z-1}}H_{0,1} - G_{-1,0}H_{1,-1} + G_{-1,\frac{z-1}{z+1}}H_{1,-1} - G_{-1,-\frac{z+1}{z-1}}H_{1,-1}\\
&+G_{0,0}H_{1,-1} + 2G_{0,1}H_{1,-1} - 2G_{0,\frac{1}{z}}H_{1,-1} - G_{0,\frac{z-1}{z+1}}H_{1,-1} - G_{0,-\frac{z+1}{z-1}}H_{1,-1} + G_{1,0}H_{1,-1}\\
&-G_{1,\frac{z-1}{z+1}}H_{1,-1} + G_{1,-\frac{z+1}{z-1}}H_{1,-1} + G_{-1,0}H_{1,0} - G_{-1,\frac{z-1}{z+1}}H_{1,0} - G_{-1,-\frac{z+1}{z-1}}H_{1,0} + G_{0,\frac{z-1}{z+1}}H_{1,0}\\
&-G_{0,-\frac{z+1}{z-1}}H_{1,0} - G_{1,0}H_{1,0} + G_{1,\frac{z-1}{z+1}}H_{1,0} + G_{1,-\frac{z+1}{z-1}}H_{1,0} - G_{-1,0}H_{1,1} + G_{-1,\frac{z-1}{z+1}}H_{1,1}\\
&-G_{-1,-\frac{z+1}{z-1}}H_{1,1} + G_{0,0}H_{1,1} + 2G_{0,1}H_{1,1} - 2G_{0,\frac{1}{z}}H_{1,1} - G_{0,\frac{z-1}{z+1}}H_{1,1} - G_{0,-\frac{z+1}{z-1}}H_{1,1}
\end{aligned}$$

- Two-loop outer mass double box
- Caron-Huot and Henn 1404.2922

1. Find a finite basis

$$I_5(s,t,u,m) = \sum_{j=1}^{\text{finite}} c_j f_j(s,t,u,m)$$

- determined by **singularities**

2. Landau bootstrap

apply enough analytic constraints to uniquely fix all $c_j$

3. Analytic regression

fit the $c_j$ numerically

# Where are the singularities?

$$I_G(p) = (n_{\rm int}-1)! \int_0^\infty \prod_{e\in E_{\rm int}(G)} \mathrm{d}\alpha_e \int \prod_{c\in\widehat{C}(G)} \mathrm{d}^d k_c \frac{1}{(\ell+i\varepsilon)^{n_{\rm int}}}\,\delta\Big(1-\sum_{e\in E_{\rm int}(G)}\alpha_e\Big)$$

A necessary condition for a singularity is that the *integrand* is singular ($\ell=0$)

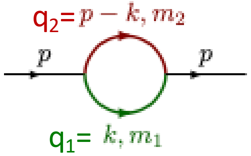
$$\boxed{\ell = \sum_{e\in E_{\rm int}(G)} \alpha_e (q_e^2 - m_e^2) \;\; =0}$$

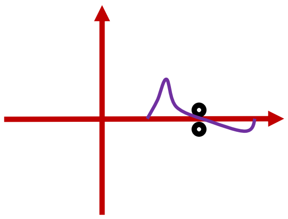
- every internal line is either on-shell ($q^2=m^2$) or $\alpha=0$ or both

A necessary condition for a singularity of the *integral* is that poles pinch the contour

Double pole:

for each loop $k_c$: $\displaystyle\sum_{e\in E_{\rm int}(G^\kappa)} \alpha_e \frac{\partial}{\partial k_c}(q_e^2 - m_e^2) = 0\,.$





integration contour
pinched between poles

- since $q_e$ are linear in $k_c$

⇒ $$\boxed{\sum_{e\,\mathrm{in\,loop}} \pm\alpha_e q_e^\mu = 0}$$ Landau loop equations

# Coleman-Norton interpretation

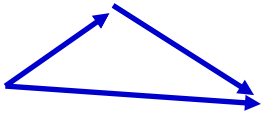
Landau equations

$$\ell = \sum_{e\in E_{\text{int}}(G)} \alpha_e(q_e^2 - m_e^2) = 0$$

$$\sum_{e\,\text{in}\,\text{loop}} \pm\alpha_e q_e^\mu = 0$$

[Coleman and Norton 1965]

4-momenta add up to zero after rescaling by α



Landau diagram is interpreted as space-time diagram

- momenta are on-shell (classical)
- $\alpha_e$ are the proper times for propagation

Gives some intuition: singularities due to classically allowed processes

# Pham interpretation

- Pham 1967
- H. Hannesdottir, A. McLeod, MDS, C. Vergu 2211.07633

Landau equations

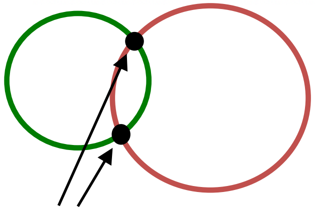
$$\ell = \sum_{e\in E_{\text{int}}(G)} \alpha_e (q_e^2 - m_e^2) = 0$$

$$\sum_{e\,\text{in}\,\text{loop}} \pm\alpha_e q_e^\mu = 0$$

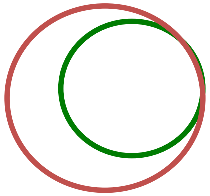
⇒ normal vectors of on-shell constraints $q^2=m^2$ are linearly dependent

on-shell constraints (Euclidean d=2)

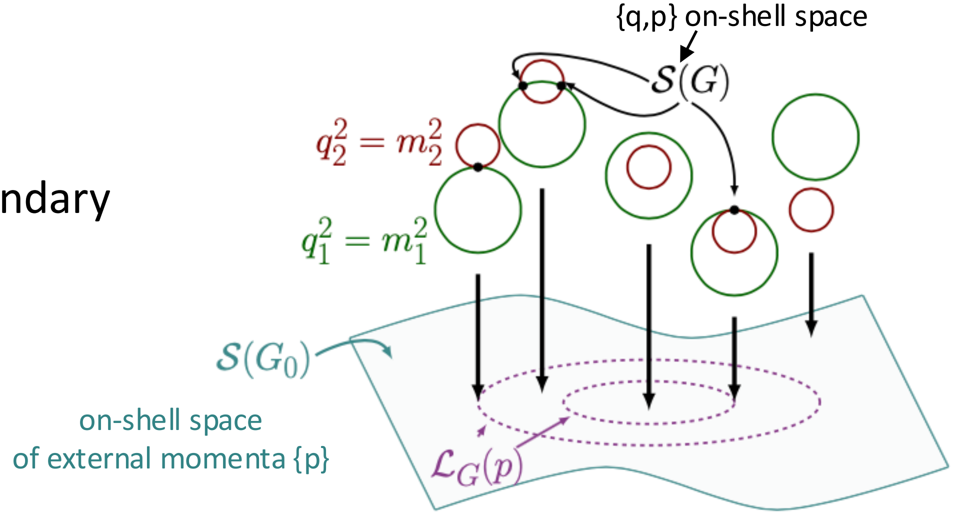
$$q_x^2 + q_y^2 = m_e^2$$



intersection satisfies both on-shell constraints



tangent on boundary of space where circles intersect

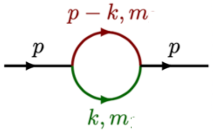


Landau variety is the boundary of the projection map

# Bubble in d=2

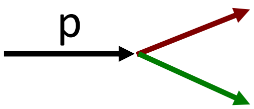
Consider the simplest 1-loop diagram: the bubble in d=2

$I_{\circ}(p) =$



$$= \int d^2k \frac{1}{k^2-m^2+i\varepsilon} \frac{1}{(p-k)^2-m^2+i\varepsilon} = \frac{-2\pi}{\sqrt{s\,(s-4m^2)}} \ln \frac{\sqrt{4m^2-s}-i\sqrt{s}}{\sqrt{4m^2-s}+i\sqrt{s}}$$

- It has a **normal threshold** branch cut starting at s=4m$^2$
  - For s > 4m$^2$ the on-shell process p → p$_1$ +p$_2$ is allowed for physical on-shell momenta



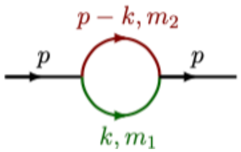
$$\text{Disc}[I(s)] = I(s+i\epsilon) - I(s-i\epsilon) = \frac{4\pi^2 i}{\sqrt{s(s-4m^2)}}$$

- There is also a **pseudthreshold** at s=0
  - There is a branch point at s=0 accessible with complex momenta
  - Does not correspond to anything physical happening

# Bubble in d=2

More general 2D bubble: generic masses

$I_{\circ}(p) =$



$$= \lim_{\varepsilon\to 0^+}\int \mathrm{d}^2 k \frac{1}{k^2-m_1^2+i\varepsilon}\,\frac{1}{(p-k)^2-m_2^2+i\varepsilon}\,,$$

Going to Feynman parameters

$$I_{\circ}(s) = \lim_{\varepsilon\to 0^+}\int_0^1 \mathrm{d}\alpha \frac{-i\pi}{s\alpha(1-\alpha)-m_1^2\alpha-m_2^2(1-\alpha)+i\varepsilon} = \ell$$

integrand is singular $(\ell = 0)$ at

$$\alpha_\pm = \frac{s+m_2^2-m_1^2 \pm \sqrt{s^2-2s(m_1^2+m_2^2)+(m_1^2-m_2^2)^2+is\varepsilon}}{2s}\,.$$

on-shell locus

- necessary but not sufficient condition for singularities of integral

singularites require pinches, i.e. $\frac{d\ell}{d\alpha} = 0$

two solutions

normal threshold $s = (m_1+m_2)^2 - i\varepsilon\,,\qquad \alpha_\pm = \frac{m_2}{m_2+m_1} + i\varepsilon\,\mathrm{sgn}(m_2-m_1)\,,$

pseudothreshold $s = (m_1-m_2)^2 + i\varepsilon\,,\qquad \alpha_\pm = \frac{m_2}{m_2-m_1} - i\varepsilon\,\mathrm{sgn}(m_2-m_1)\,.$

- location of branch points
- solutions to Landau equations

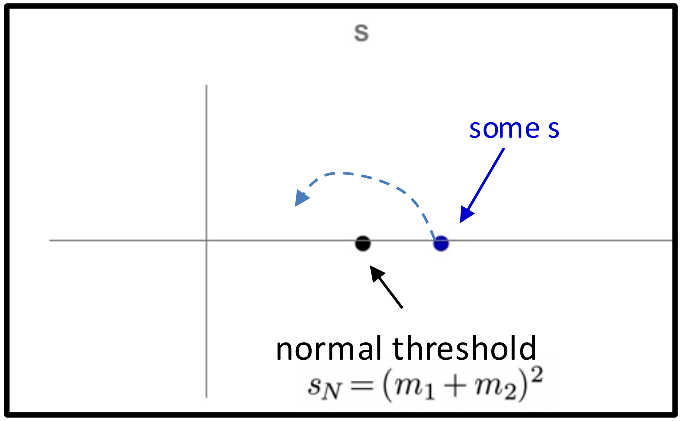
# Homotopy and Homology

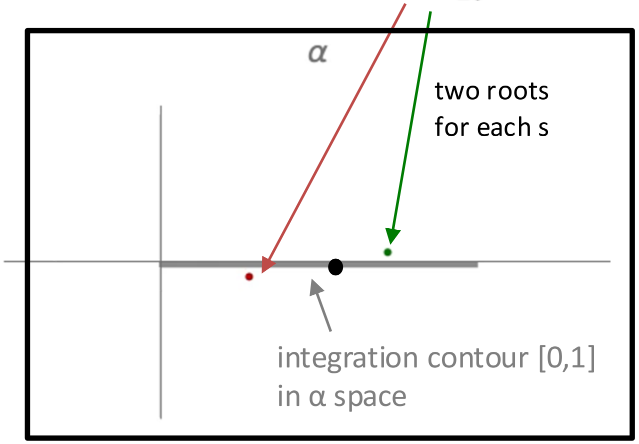
$$I_{\bigcirc}(s) = \lim_{\varepsilon\to 0^+}\int_0^1 \mathrm{d}\alpha \frac{-i\pi}{s\alpha(1-\alpha)-m_1^2\alpha-m_2^2(1-\alpha)+i\varepsilon}$$

$$= \int_0^1 \frac{d\alpha}{[\alpha-\alpha_+(s)][\alpha-\alpha_-(s)]}$$

on-shell locus: $\alpha = \alpha_\pm$

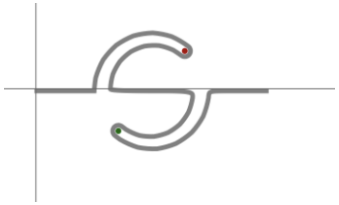
$$\alpha_\pm = \frac{s+m_2^2-m_1^2 \pm \sqrt{s^2-2s(m_1^2+m_2^2)+(m_1^2-m_2^2)^2+is\varepsilon}}{2s}.$$

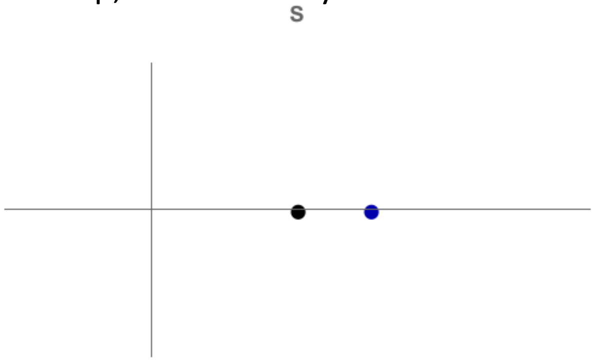


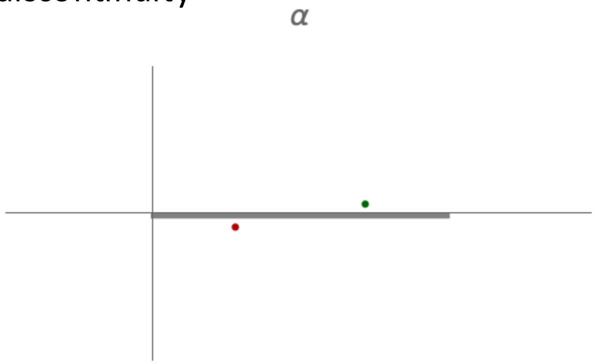


What happens as we take a monodromy (discontinuity) of s around $s_N$?

- Poles $\alpha_\pm$ move around too
- Contour must move out of the way to avoid poles
- After full loop, can use Leray residue theorem to get discontinuity







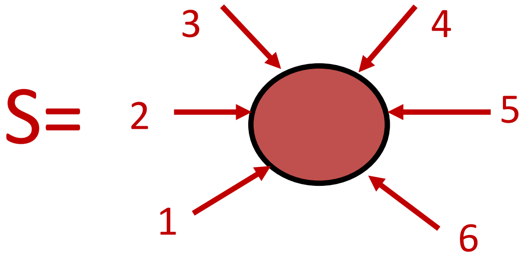
# Sequential discontinuities

- Can take a discontinuity of a discontuitty

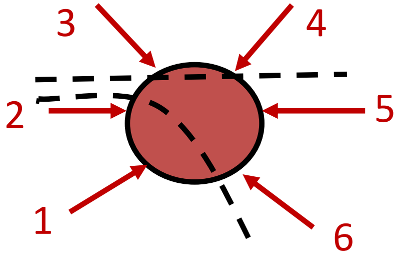
$$\mathrm{Disc}_N\,\mathrm{Disc}_P\,I = \frac{-8\pi^2 i}{\sqrt{(s-s_N)(s-s_P)}}$$

**Steinman relations :** S matrix cannot have sequential discontinuities in crossed channels

[Steinmann 1960]

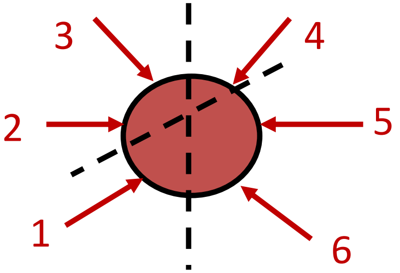


uncrossed



possible term: $\ln(p_1+p_2)^2 \ln(p_3+p_4)^2$

crossed

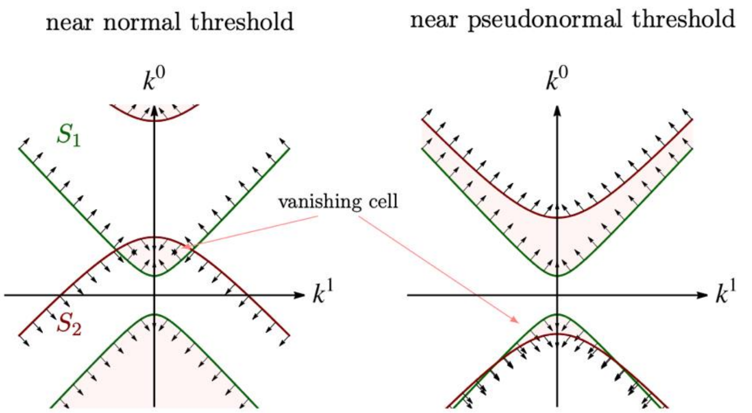


cannot appear $\ln(p_1+p_2+p_3)^2 \ln(p_2+p_3+p_4)^2$

- Algebraic geometry need to understand contour deformations and pinches
- Gets pretty complicated...

# Different kinds of singularities

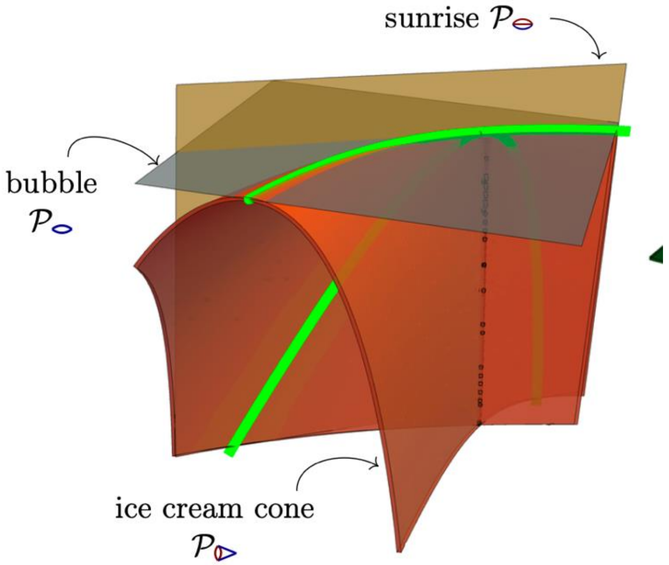
Simple pinches



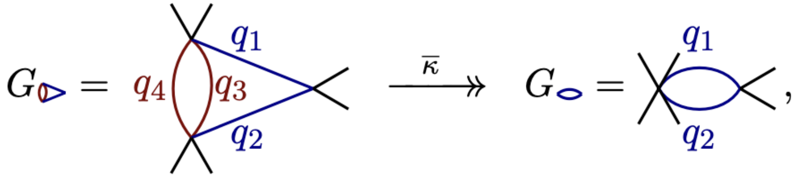
- hypersufaces meet transversely
- e.g. physical thresholds

McLeod, Hannesdottir, MDS, Vergu
arXiv:2211.07633

Non-simple pinches


sunrise $\mathcal{P}_{\ominus}$
bubble
$\mathcal{P}_{\circ}$
ice cream cone
$\mathcal{P}_{\triangleright}$

- hypersufaces meet tangentially
    - e.g. sunrise in the ice-cream cone



- Permanent pinches (e.g. IR divergences)
- Pinches at infinity

# Symbol

The symbol is an efficient way to represent polylogarithmic Feynman integrals

[Goncharov, Spradlin, Vergu, Volovich PRL 2010]

- Used symbol to simplify 17 page 2-loop 6 point amplitude to a few lines

15th anniversary!

Polylogarithms are iterated integrals

$$\mathrm{Li}_1(s) \equiv \int_0^s \frac{dx}{1-x} = -\ln(1-s) \qquad \mathrm{Li}_2(s) \equiv \int_0^s \frac{dx}{x}\mathrm{Li}_1(s)$$

etc.

Symbol is a map that extracts the dlog forms

$$\mathcal{S}\left[\int_a^b d\ln R_1 \circ \cdots \circ d\ln R_n\right] = R_1 \otimes \cdots \otimes R_n \qquad \mathcal{S}\left[\ln x \ln y\right] = x \otimes y + y \otimes x$$

$$\mathcal{S}\left[\mathrm{Li}_n(x)\right] = -(1-x) \otimes \underbrace{x \otimes \cdots \otimes x}_{n-1} \qquad \mathcal{S}\left[\frac{1}{n!}\ln^n x\right] = \underbrace{x \otimes \cdots \otimes x}_{n}$$

## Advantages

- Efficient way to represent integrals
- Possible logarithmic singularities ⇔ symbol “letters” vanish
- Transparently encodes discontinuities and sequential discontinties

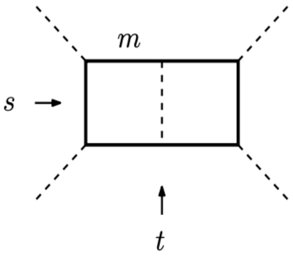
# Constructing a Finite basis

The **symbol** is the key to making the the problem finite

## 1. Identify singularities

- All solution to Landau equations
- Some singularities may be on higher Riemann sheets

outer mass double box



physical singularities

$$s = 4m^2, \quad s \to \infty,$$
$$t = 4m^2, \quad t \to \infty,$$
$$m^2 = 0,$$

unphysical singularities

$$s = 0, \qquad t = 0, \qquad s + t = 0\,,$$
$$st + 4m^2 s + 4m^2 t = 0\,.$$

$$u = -\frac{4m^2}{s}\,, \qquad v = -\frac{4m^2}{t}\,.$$

## 2. Identify possible letters

- algebraic functions (with square roots) of singularities with no new singularities

$$\beta_u = \sqrt{1+u}\,, \qquad \beta_v = \sqrt{1+v}\,,$$
$$\beta_{uv} = \sqrt{1+u+v}\,.$$

$$L_1 = u, \qquad L_2 = v, \qquad L_7 = \frac{\beta_v - 1}{\beta_v + 1}, \qquad L_8 = \frac{\beta_{uv} - 1}{\beta_{uv} + 1},$$
$$L_3 = 1 + u, \qquad L_4 = 1 + v, \qquad L_9 = \frac{\beta_{uv} - \beta_u}{\beta_{uv} + \beta_u}, \qquad L_{10} = \frac{\beta_{uv} - \beta_v}{\beta_{uv} + \beta_v}\,,$$
$$L_5 = u + v, \qquad L_6 = \frac{\beta_u - 1}{\beta_u + 1}, \qquad L_{11} = 1 + u + v\,,$$

## 3. Finite basis

- sum over all possible symbols using the alphabet

$$\mathcal{S}[\mathcal{I}] = \sum c_{\{i\}} L_{i_1} \otimes L_{i_2} \otimes L_{i_3} \otimes L_{i_4}$$

# Landau Bootstrap

MDS, Hannesdottir, McLeod, Vergu 2410.02424

A. Construct finite basis

1. Find singularities
2. Find letters
3. Construct finite basis

$$\mathcal{S}[\mathcal{I}] = \sum c_{\{i\}} L_{i_1} \otimes L_{i_2} \otimes L_{i_3} \otimes L_{i_4}$$

B. Fix the coefficients

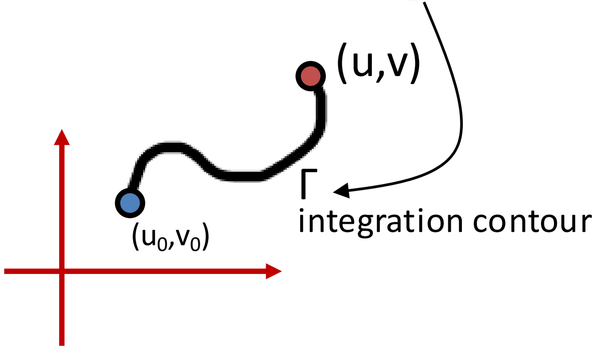
- Integrability
- Galois symmetry
- α-positivity
- First-entry conditions
- Last-entry conditions
- Geneological constraints
  - Sequential discontinuities
  - Cluster adjacency conditions
- Regions analysis
- Direct calculation

C. Fix the non-symbol rational prefactor

# Integrability

- Every interated dlog integral has a symbol

$$f(u,v;\Gamma) = c_{i_1,i_2,\ldots,i_n} \int_\Gamma d\ln L_{i_1} \circ \cdots \circ d\ln L_{i_n} \,. \quad \underset{\longleftarrow}{\longrightarrow} \quad \sum c_{i_1,i_2,\ldots,i_n} L_{i_1} \otimes \cdots \otimes L_{i_n}$$



- Not every symbol corresponds to a function

**Learning the Simplicity of Scattering Amplitudes**

Clifford Cheung (Caltech), Aurélien Dersy (Harvard U. and IAIFI, Cambridge), Matthew D. Schwartz (Harvard U. and IAIFI, Cambridge) (Aug 8, 2024)

e-Print: 2408.04720 [hep-th]

- For f(u,v,Γ) to be a function, must be independent of local path deformations

$$\Rightarrow \quad [\partial_u, \partial_v] f = 0$$

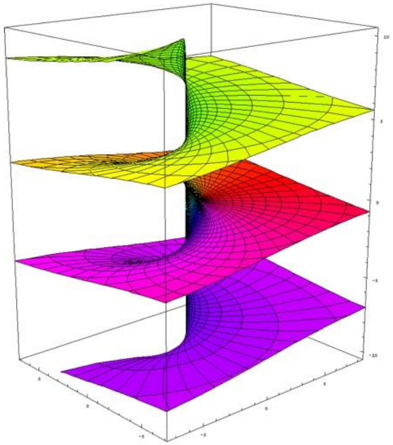
- Derivatives only act on the last entry of the symbol (end of integration contour)

$$\partial_u[S\otimes K\otimes L] = (\partial_u \ln L)[S\otimes K]$$

$$\partial_v\partial_u[S\otimes K\otimes L] = (\partial_v\partial_u \ln L)[S\otimes K] + (\partial_u \ln L)(\partial_v \ln K)S$$

$$[\partial_u,\partial_v][S\otimes K\otimes L] = \underbrace{[(\partial_u \ln L)(\partial_v \ln K) - (\partial_v \ln K)(\partial_u \ln L)]}_{\substack{\text{must vanish} \\ \text{(integrability condition)}}} S$$

# α positivity



- Symbol encodes all branch points, even on unphysical sheets

$$I_G(p) = (n_{\mathrm{int}} - 1)! \int_0^\infty \prod_{e \in E_{\mathrm{int}}(G)} \mathrm{d}\alpha_e \int \prod_{c \in \widehat{C}(G)} \mathrm{d}^d k_c \frac{1}{(\ell + i\varepsilon)^{n_{\mathrm{int}}}} \delta\Big(1 - \sum_{e \in E_{\mathrm{int}}(G)} \alpha_e\Big)$$

actual Feynman integral on physical sheet is over positive α

- Discontinuities/monodromies act on first entry of the symbol

$$I = \int \omega_1 \int \omega_2 \ldots \int \omega_n$$

$$dI = \omega_1 \int \omega_2 \ldots \int \omega_n$$

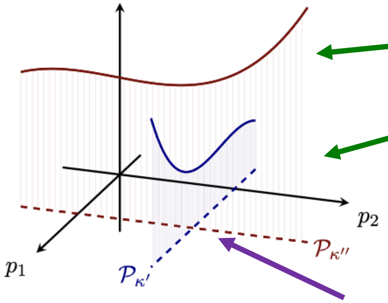
- Singularity for physical momenta (physical sheet) ↔ singularity of first entry
  - See if α > 0 in solutions to Landau equations: constrain first symbol entries

# Genealogical constraints

Which symbol entries can be adjacent?

- Adjacent means sequential discontinuities are possible

Pham approach



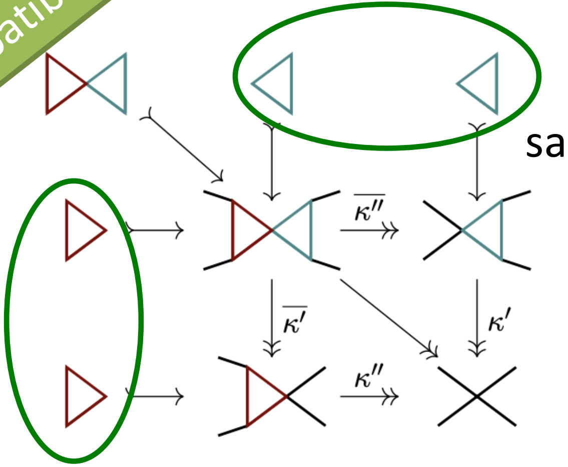
on-shell surfaces may not intersect in *internal momenta*

- vanishing cell from first monodromy doesn't intersect integration contour of second
- sequential monodromy vanishes

$$\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa'}}\right)\left(\mathbb{1}-\mathscr{M}_{\mathcal{P}_{\kappa''}}\right)I_G(p)=0$$

Singular surfaces intersect transversaly in external momenta

compatible

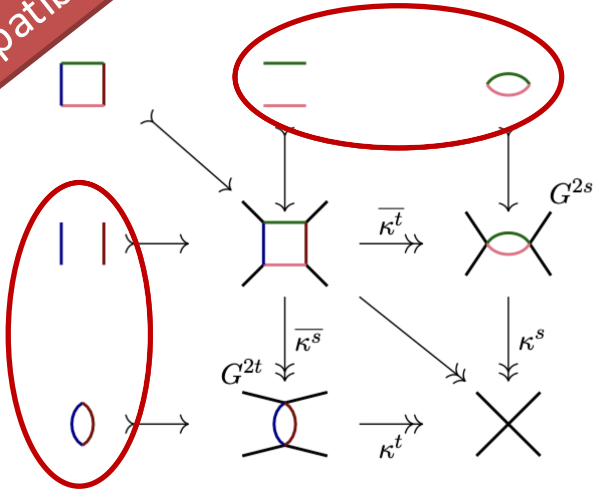


same

same

incompatible

different

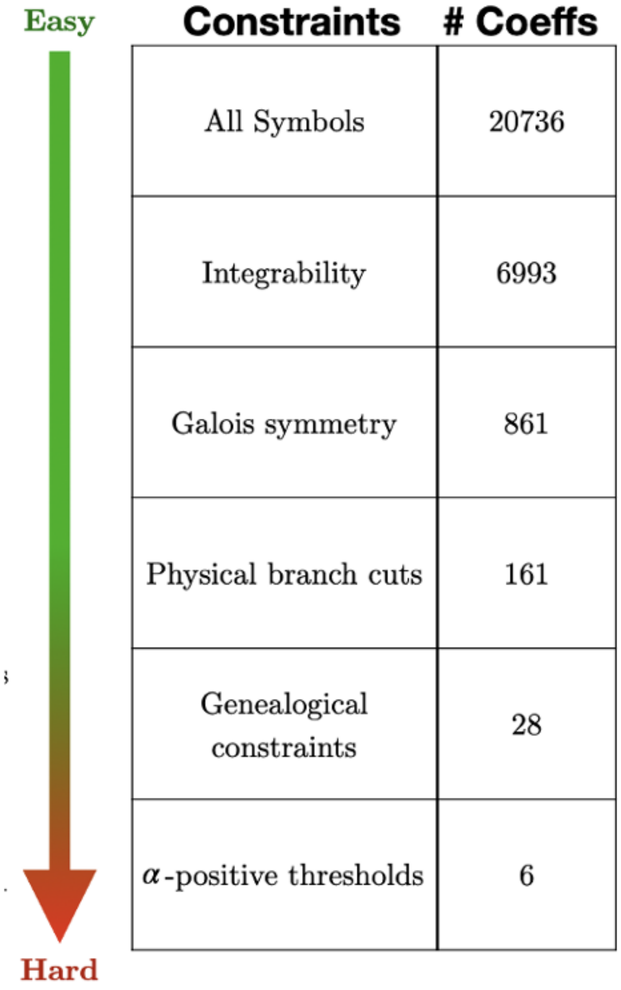


different

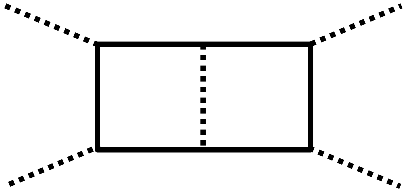
- e.g. Steinmann relations

# Outer-mass double box

Landau boostrap

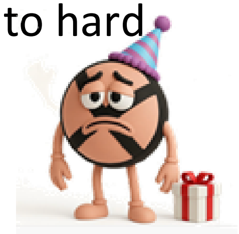


| Constraints | # Coeffs |
|---|---|
| All Symbols | 20736 |
| Integrability | 6993 |
| Galois symmetry | 861 |
| Physical branch cuts | 161 |
| Genealogical constraints | 28 |
| $\alpha$-positive thresholds | 6 |



Caron-Huot & Henn ‘14

- Weight 4 symbol, 12 Letters
- Work from easy generic stuff (integrability) to hard



$$
\begin{aligned}
\mathcal{S}(\tilde{\mathcal{I}}_{\text{dbox}}) = {} & -L_6 \otimes \frac{L_1}{L_3} \otimes L_6 \otimes L_9 - L_6 \otimes \frac{L_1}{L_3} \otimes L_9 \otimes L_6 \\
& + L_6 \otimes L_6 \otimes \frac{L_1 L_2}{L_3 L_5} \otimes L_9 + L_6 \otimes L_9 \otimes \frac{L_2}{L_5} \otimes L_6 \\
& + L_6 \otimes L_6 \otimes L_8 \otimes L_6 + L_6 \otimes L_9 \otimes L_8 \otimes L_9 \\
& + L_7 \otimes L_{10} \otimes \frac{L_2}{L_5} \otimes L_6 + L_7 \otimes L_{10} \otimes L_8 \otimes L_9 \\
& + L_7 \otimes L_7 \otimes \frac{L_1}{L_5} \otimes L_9 + L_7 \otimes L_7 \otimes L_8 \otimes L_6 \,.
\end{aligned}
$$

- Agrees with Caron-Huot & Henn ’14
- Also need rational prefactor -- not in the sybmol
  - Compute from maximal cut

Landau bootstrap works!

# Top down or bottom up?

1. Top down:
Landau bootstrap

Easy

Hard

| Constraints | # Coeffs |
|---|---|
| All Symbols | 20736 |
| Integrability | 6993 |
| Galois symmetry | 861 |
| Physical branch cuts | 161 |
| Genealogical constraints | 28 |
| $\alpha$-positive thresholds | 6 |

Hard

Easy

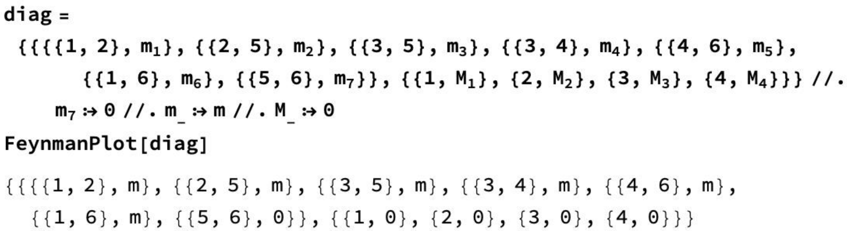
2. Bottom up:
Analytic regression with
lattice reduction

# Solving the Landau equations

Lots of ways to solve the Landau equations

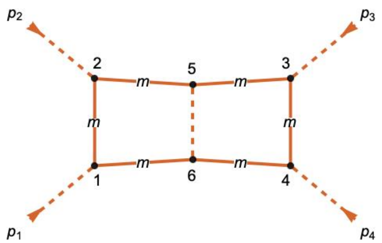
- Solve them by hand (e.g. Eden et al 1950)
- HyperInt (Panzer 2014)
- PLD (Fevola, Mizera, Telen 2013)
- BaikovLetter (Jiang et al 2024
- Recursive approach (Caron-Huot, Correia and Giroux 2024)
- Numerical implementation for any diagram (Correia, Giroux, Mizera 2024) SOFIA

Input

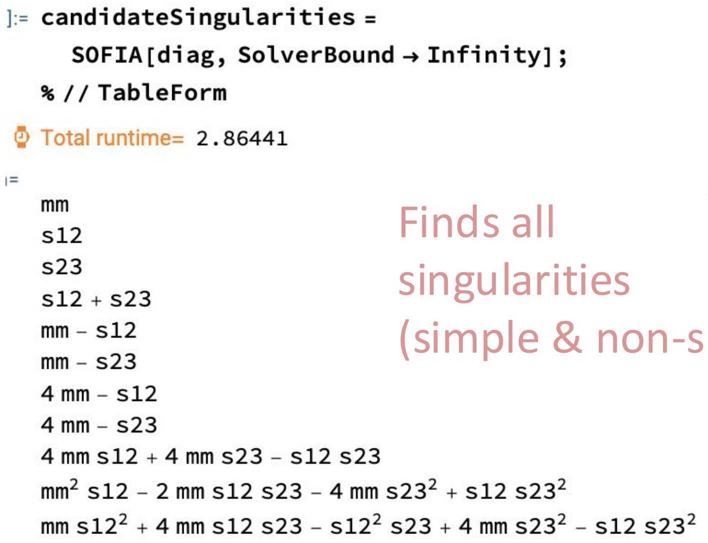


```
diag =
 {{{{1, 2}, m1}, {{2, 5}, m2}, {{3, 5}, m3}, {{3, 4}, m4}, {{4, 6}, m5},
     {{1, 6}, m6}, {{5, 6}, m7}}, {{1, M1}, {2, M2}, {3, M3}, {4, M4}}} //.
   m7 :→ 0 //. m_ :→ m //. M_ :→ 0
FeynmanPlot[diag]

{{{{1, 2}, m}, {{2, 5}, m}, {{3, 5}, m}, {{3, 4}, m}, {{4, 6}, m},
  {{1, 6}, m}, {{5, 6}, 0}}, {{1, 0}, {2, 0}, {3, 0}, {4, 0}}}
```



Output



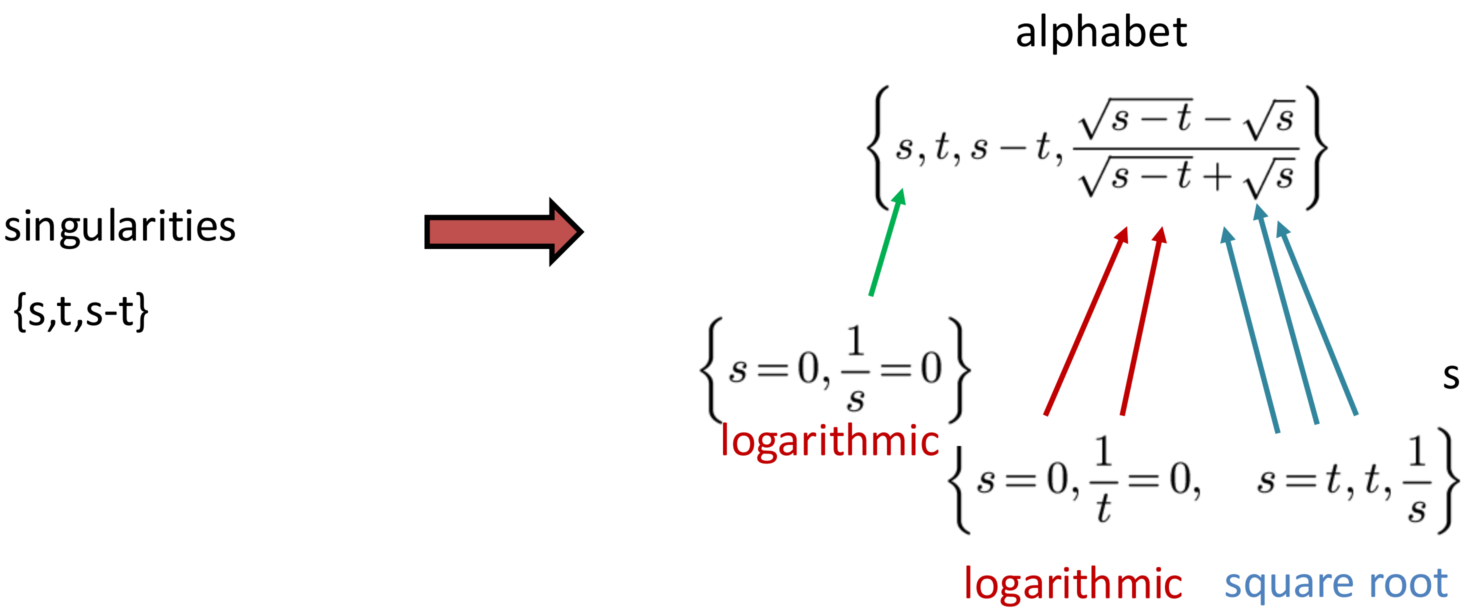
```
]:= candidateSingularities =
      SOFIA[diag, SolverBound → Infinity];
    % // TableForm

    Total runtime= 2.86441

//=
    mm
    s12
    s23
    s12 + s23
    mm - s12
    mm - s23
    4 mm - s12
    4 mm - s23
    4 mm s12 + 4 mm s23 - s12 s23
    mm^2 s12 - 2 mm s12 s23 - 4 mm s23^2 + s12 s23^2
    mm s12^2 + 4 mm s12 s23 - s12^2 s23 + 4 mm s23^2 - s12 s23^2
```

Finds all singularities (simple & non-simple)

# Singularities to letters



singularities

{s,t,s-t}

alphabet

$$\left\{ s, t, s-t, \frac{\sqrt{s-t}-\sqrt{s}}{\sqrt{s-t}+\sqrt{s}} \right\}$$

$$\left\{ s=0, \frac{1}{s}=0 \right\}$$

logarithmic

s

$$\left\{ s=0, \frac{1}{t}=0, \quad s=t, t, \frac{1}{s} \right\}$$

logarithmic square root

Landau’s original paper determined if physical singularities were logarithmic or square-root

- In general, singularities may appear multiple times on multiple sheets
- Still true that singuarites are always either logarithmic or square root
- Automated codes like SOFIA seem to be able to tell logarithmic from square root

# Finite basis

Location of Landau singularities

$$\begin{gathered} s = 4m^2, \quad s \to \infty, \\ t = 4m^2, \quad t \to \infty, \\ m^2 = 0, \\ s = 0, \qquad t = 0, \qquad m^2 \to \infty\,, \\ s + t = 0, \qquad st - 4m^2 s - 4m^2 t = 0\,. \end{gathered}$$

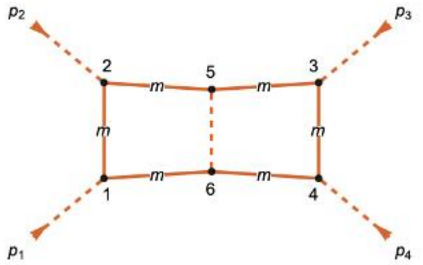
symbol alphabet

$$\tilde{A} = \Big\{ u, v, 1+u, 1+v, u+v, 1+u+v, \frac{\beta_u - 1}{\beta_u + 1}, \frac{\beta_v - 1}{\beta_v + 1}, \frac{\beta_{uv} - 1}{\beta_{uv} + 1}, \frac{\beta_{uv} - \beta_u}{\beta_{uv} + \beta_u}, \frac{\beta_{uv} - \beta_v}{\beta_{uv} + \beta_v}, \frac{\beta_{uv} - \beta_u \beta_v}{\beta_{uv} + \beta_u \beta_v} \Big\}$$

$$\beta_u = \sqrt{1+u}, \; \beta_v = \sqrt{1+v} \;; \beta_{uv} = \sqrt{1+u+v}.$$

- SOFIA can also produce the alphabet (may be larger than needed)

- Symbol weight <= 2L, L = loop order



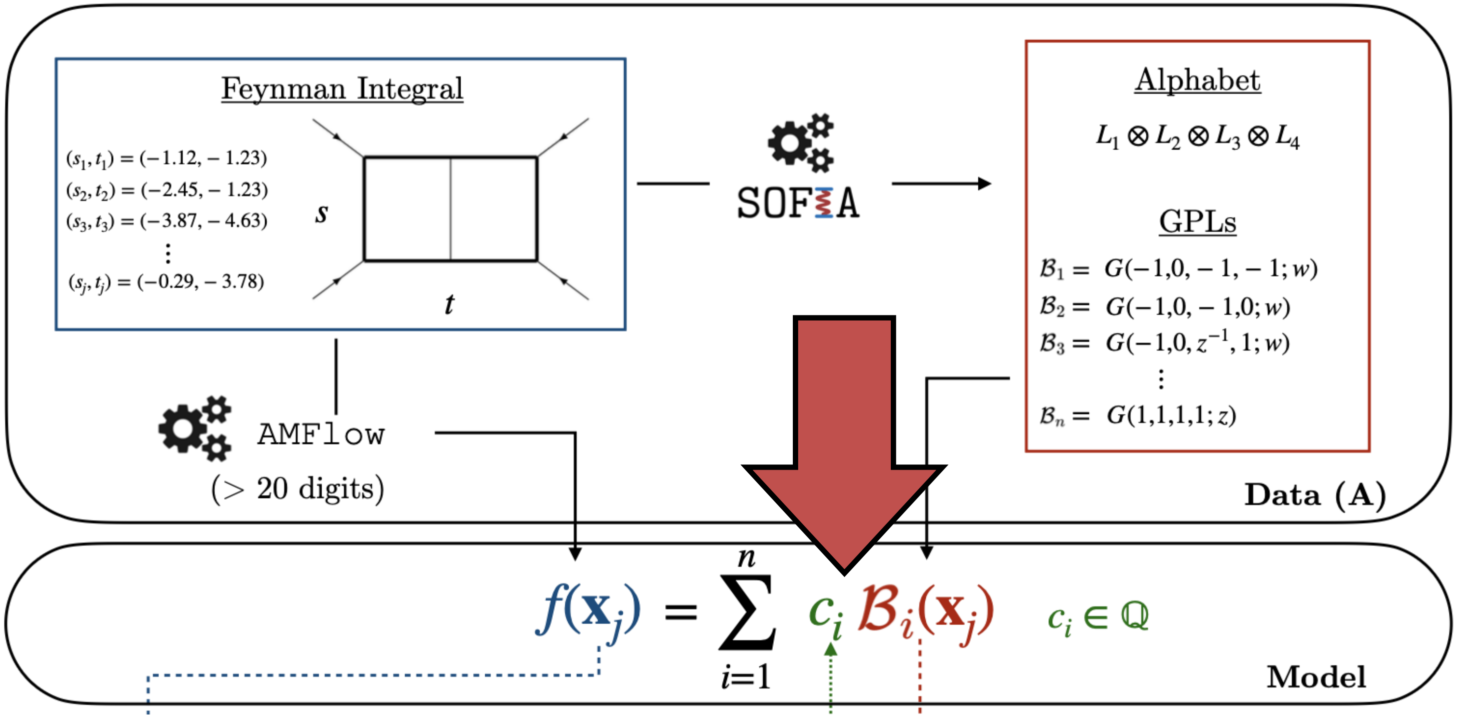
2 loop outer-mass double box

- 12 letters
- $12^4$ = 20,736 symbol entries

$$\mathcal{S}[\mathcal{I}] = \sum c_{\{i\}} L_{i_1} \otimes L_{i_2} \otimes L_{i_3} \otimes L_{i_4}$$

Finite basis!

# Numerical bootstrap



Feynman Integral

$(s_1,t_1) = (-1.12,-1.23)$

$(s_2,t_2) = (-2.45,-1.23)$

$(s_3,t_3) = (-3.87,-4.63)$

$\vdots$

$(s_j,t_j) = (-0.29,-3.78)$

$s$

$t$

SOFIA

Alphabet

$L_1 \otimes L_2 \otimes L_3 \otimes L_4$

GPLs

$\mathcal{B}_1 = G(-1,0,-1,-1;w)$

$\mathcal{B}_2 = G(-1,0,-1,0;w)$

$\mathcal{B}_3 = G(-1,0,z^{-1},1;w)$

$\vdots$

$\mathcal{B}_n = G(1,1,1,1;z)$

AMFlow

($>$ 20 digits)

Data (A)

$$f(\mathbf{x}_j) = \sum_{i=1}^{n} c_i\, \mathcal{B}_i(\mathbf{x}_j) \qquad c_i \in \mathbb{Q}$$

Model

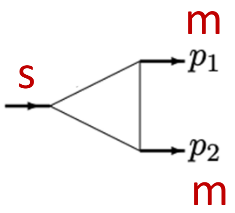
Start with finite basis

can we just fit the $c_j$ numerically?

Requirements

1. Evaluate f(x) numerically to high precision
2. Evaluate B(x) to high precision
3. Solve the equation without losing precision

# 1-loop triangle



exact answer: $T = \quad = \frac{x}{m^2(1-x^2)}\left(-\frac{\pi^2}{3} + 4G(0,-1,x) - 2G(0,0;x)\right)$

one invariant $x = \frac{1-\sqrt{1+\frac{4m^2}{s}}}{1+\sqrt{1+\frac{4m^2}{s}}}$

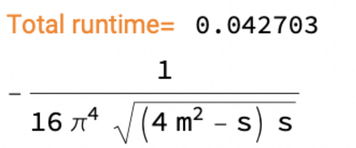
Alphabet (from Sofia)

3 letters

$$A = \{L\} = \{m, s, 4m^2 - s\} \longleftrightarrow \{x, 1+x, 1-x\}$$

rational prefactor

(Sofia computes)



Total runtime= 0.042703

$$-\frac{1}{16\,\pi^4\,\sqrt{(4\,m^2 - s)\,s}}$$

Basis is

$$T(x) = \sum_{i=1}^{17} c_i\mathcal{B}_i(x) = P(x) \times [\underbrace{c_{ij}L_i \otimes L_j}_{\text{9 terms weight 2}} + \underbrace{(a_j + b_j\pi)L_j}_{\text{6 terms weight 1}} + \underbrace{d_1\pi^2 + d_2\zeta_3}_{\text{2 numbers}}]$$

17 constants

- Need T(x) and $B_i$(x) to high precision

# Numerical Feynman Integration

- AMFlow: numerical evaluation for Feynman loop integrals to high precision

Integration-by-Parts (IBP) [Chetyrkin, Tkachov, 1981]

Rational functions

Master integrals (basis)

$$\mathcal{I}(p_i, m_e) = \int \prod_{j=1}^{L} \mathrm{d}^D k_j \frac{N(p_i, k_j)}{\prod_{e=1}^{E}(q_e^2 - m_e^2 + i\varepsilon)} \Rightarrow \mathcal{I} = \sum_a R_a(p_i \cdot p_j, m_e)\, \mathrm{MI}_a$$

Auxiliary mass flow method (AMFLow) [Liu, Ma, 1801.10523; 2107.01864; 2201.11669; 2201.11637]

1. Introduce an auxiliary mass η to some of the propagator denominators
2. Set up closed differential equations w.r.t η using IBPs
3. Solve the differential equations numerically with boundary conditions $\eta \rightarrow \infty$

Timing on 1-loop triangle

- each phase space point takes 5-10 CPU-min for 30 signficiant digits

# Numerical basis functions

$$T(x) = \sum_{i=1}^{17} c_i \mathcal{B}_i(x) = P(x) \times [c_{ij} L_i \otimes L_j + (a_j + b_j \pi) L_j + d_1 \pi^2 + d_2 \zeta_3]$$

- Rational prefactor evaluation is instant
- Weight-2 symbols can be integrated into closed form expressions
  - FiberSymbol in Polylogtools can do this
  - Numerical evaluation is very fast

Integrals of highter weight symbols are not always known

- Can always integrate numerically along a path

$$\tilde{B} = \sum_{i=1}^{|\tilde{A}|} c_{i_1,i_2,i_3,i_4} \int_0^1 d\log L_{i_4}(\lambda_4) \int_0^{\lambda_4} d\log L_{i_3}(\lambda_3) \int_0^{\lambda_3} d\log L_{i_2}(\lambda_2) \int_0^{\lambda_2} d\log L_{i_1}(\lambda_1)\,,$$

- Individual terms may be path dependent, but final result is integrable
- Can do the first and last integral ananlytically
  - Reduce weight by 2: speeds up integration tremendously
- Sometimes analytic integrals can be so complicated that it is faster to do the integral numerically
  - More work needed on efficient numerical evaluation

# Matrix inversion

- Evaluate both sides of this equation at 17 points
- Solve 17 linear equations for coefficients
  - i.e. invert the matrix

$$\boxed{T(x) = \sum_{i=1}^{17} c_i \mathcal{B}_i(x)}$$

$$M_{ij} = \mathcal{B}_i(\mathbf{x}_j) \quad \Rightarrow \quad c_i = (M^{-1})_{\mathrm{ij}} \cdot T(x_j)$$

Problems:

1. No way to impose that $c_j$ are rational
   - Cannot ever resolve functions that differ by irrational constants

$$2\,[\ln x] + 3\,[\pi \ln x] = (1+\pi)[\ln x] + \left(2 + \frac{1}{\pi}\right)[\pi \ln x]$$

   - Even with infinite precision, will not find the exact analytic answer

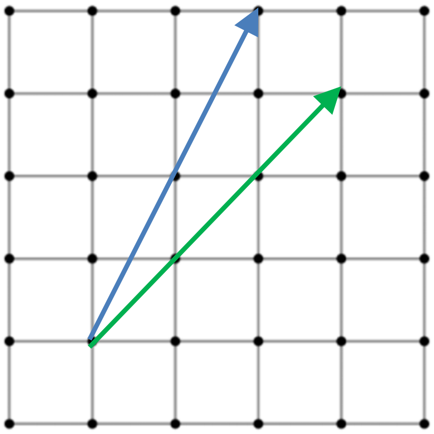
2. Matrix inversion loses precision very fast
   - Controlled by condition number

$$\kappa(M) = \|M\| \|M^{-1}\| \,.$$

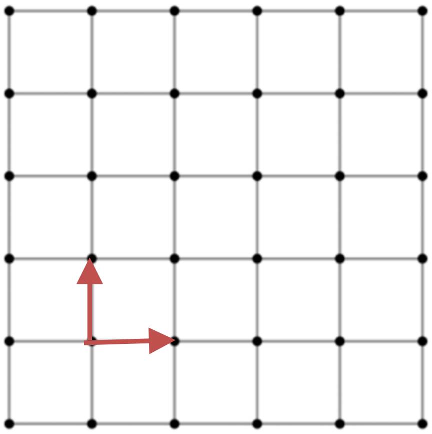
   - Generally our matrices will be ill-conditioned
     (they come from smooth functions)

# Lattice reduction

vectors span a lattice



$$\vec{u}_1^2 + \vec{u}_2^2 = 24$$



$$\vec{v}_1^2 + \vec{v}_2^2 = 2$$

- Lattice reduction finds another set of vectors for same lattice
- Can minimize some norm (length of lattice vectors)
- NP-Hard problem: no polynomial-time algorithm for truly best solution
- Efficient algorithms exist to find what is almost always the minimum

# Lattice reduction

Rational number coefficients can be fit for numbers using lattice reduction

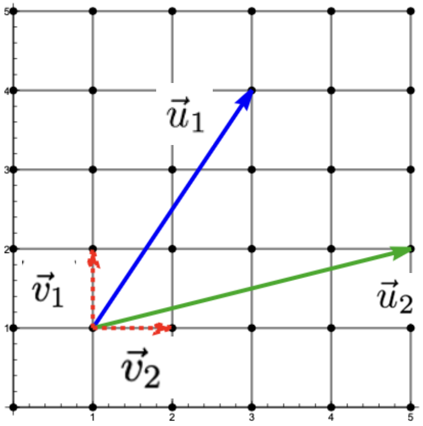
$$f = -\int_0^1 du \int_0^1 dv \frac{\log(1-uv) + v\log(1-u)}{uv} = \frac{\pi^2}{6} + \zeta_3 \,. = c_1\pi^2 + c_2\zeta_3 = \vec{c}\cdot\vec{v}$$

f=2.847 ⟷ Assume 4 digits known ⟷ $\pi^2$=9.870 $\zeta_3$=1.202

Multiply by $10^3$ and put into a matrix

$$\begin{pmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vec{u}_3 \end{pmatrix} = \begin{pmatrix} 10^3 f & 1 & & \\ 10^3\pi^2 & & 1 & \\ 10^3\zeta_3 & & & 1 \end{pmatrix} = \begin{pmatrix} 2847 & 1 & 0 & 0 \\ 9870 & 0 & 1 & 0 \\ 1202 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[\text{lattice reduction}]{} \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{pmatrix} = \begin{pmatrix} 0 & -6 & 1 & 6 \\ 3 & -11 & 5 & -15 \\ 62 & 4 & -2 & 7 \end{pmatrix}$$

vectors span
a lattice



- Lattices are the same so v in the span of u

$$\vec{v}_1 = -6\vec{u}_1 + \vec{u}_2 + 6\vec{u}_3 .$$

First component

$$0 = 10^3 \times (-6f + \pi^2 + 6\zeta^3)$$ ✓

# Precision requirements

Rational number coefficients can be fit for numbers lattice reduction

$$f = -\int_0^1 du \int_0^1 dv \frac{\log(1-uv) + v\log(1-u)}{uv} = \frac{\pi^2}{6} + \zeta_3 = c_1\pi^2 + c_2\zeta_3 = \vec{c}\cdot\vec{v}$$

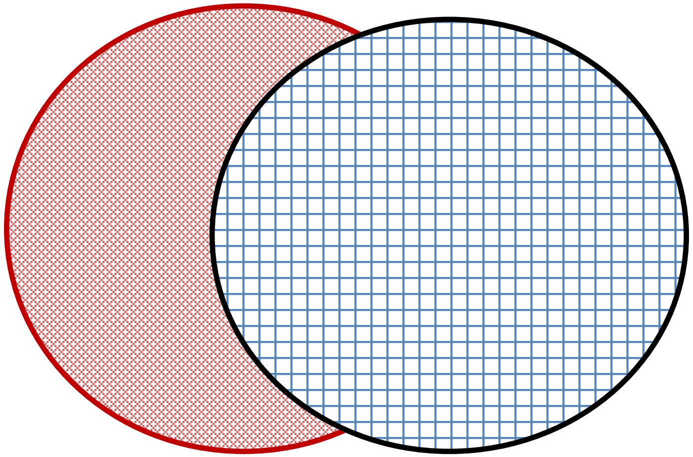
Q: how many digits of f are required to fit rational $c_1$ and $c_2$?

$$\vec{v} = (\pi^2, \zeta_3)$$

assume size of c's

$$c_1 \sim c_2 \lesssim 10^R$$

- Multiple solutions imples that $\vec{c}\cdot\vec{v} = 0$
- Multiply by GCD so $c_1$ and $c_2$ are integers
  - There are $(10^R)^n = 10^{2R}$ vectors $(c_1,c_2)$
- Assume d digits of precision on v
  - $\vec{c}\cdot\vec{v}$ produce $10^{2R}$ d digit numbers
- There are only $10^d$ d-digit numbers in all
- Need precision $d > nR$ to fit pure numbers
  - Information content must be sufficient



# Precision requirements

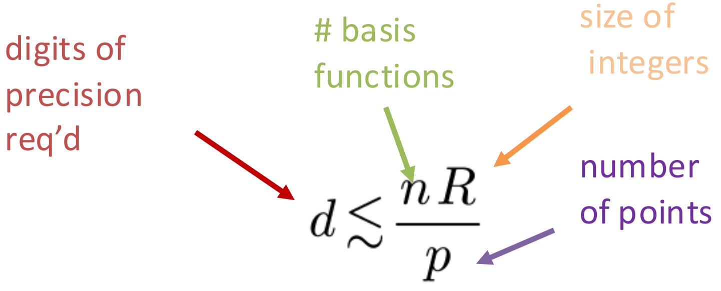
For fitting *functions* we can sample at multiple points

$$f(x) = G(0,1;x) - G(1,-1;x) \qquad x_1 = 4/10, x_2 = 9/10$$

$$\mathcal{B}(x) = \{G(1,0;x), G(0,1;x), G(0,-1;x), G(1,-1;x)\}$$

$$M = \text{round}\, 10^s \left( \begin{array}{c|c} f(\mathbf{x_1})\cdots f(\mathbf{x_p}) & 10^{-s}\,\mathbb{I}_{n+1} \\ \mathcal{B}_1(\mathbf{x_1})\cdots \mathcal{B}_1(\mathbf{x_p}) & \\ \vdots & \\ \mathcal{B}_n(\mathbf{x_1})\cdots \mathcal{B}_n(\mathbf{x_p}) & \end{array} \right) = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \\ \mathbf{v}_5 \end{pmatrix} = \begin{pmatrix} -35 & -24 & 1 & 0 & 0 & 0 & 0 \\ 92 & 154 & 0 & 1 & 0 & 0 & 0 \\ -45 & -129 & 0 & 0 & 1 & 0 & 0 \\ 36 & 75 & 0 & 0 & 0 & 1 & 0 \\ -10 & -106 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- With p points and d digits
  - net digits of information is p × d
- Expected digits needed



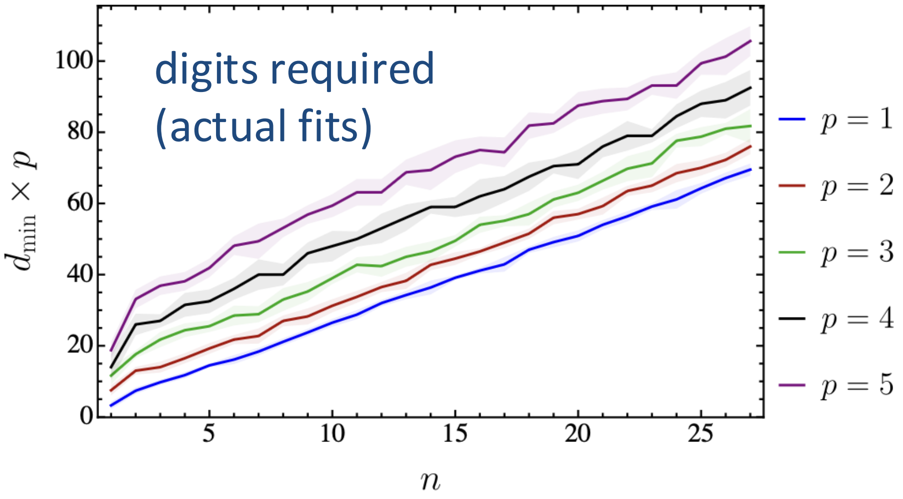
digits of precision req'd

# basis functions

size of integers

number of points

$$d \lesssim \frac{nR}{p}$$



digits required (actual fits)

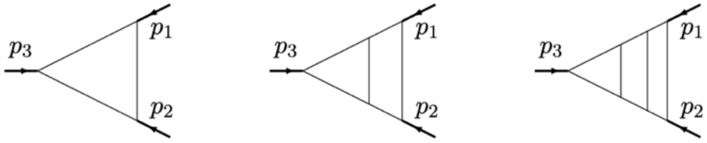
number of functions

We can trade off digits of precision for number of points

- Will work even if only a few digits of precision are available!

# Example 1: Triangles

Triangle ladder diagrams



$$z\bar{z} = p_2^2/p_1^2,$$
$$(1-z)(1-\bar{z}) = p_3^2/p_1^2$$

exact results known

$$T_1(z) = \frac{1}{z-\bar{z}}\left[2\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \log(z\bar{z})\log\left(\frac{1-z}{1-\bar{z}}\right)\right],$$
$$T_2(z) = \frac{1}{(1-z)(1-\bar{z})(z-\bar{z})}\Big[6\mathrm{Li}_4(z) - 6\mathrm{Li}_4(\bar{z}) - 3\log(z\bar{z})\left(\mathrm{Li}_3(z) - \mathrm{Li}_3(\bar{z})\right)$$
$$+\frac{1}{2}\log^2(z\bar{z})\left(\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})\right)\Big],$$
$$T_3(z) = \frac{1}{(1-z)^2(1-\bar{z})^2(z-\bar{z})}\Big[20\mathrm{Li}_6(z) - 20\mathrm{Li}_6(\bar{z}) - 10\log(z\bar{z})\left(\mathrm{Li}_5(z) - \mathrm{Li}_5(\bar{z})\right)$$
$$+\log^2(z\bar{z})\left(\mathrm{Li}_4(z) - \mathrm{Li}_4(\bar{z})\right) - \frac{1}{6}\log^3(z\bar{z})\left(\mathrm{Li}_3(z) - \mathrm{Li}_3(\bar{z})\right)\Big]$$

Full alphabet (from SOFIA)

$$A_{1,2} = \left\{ z\bar{z},\ (1-z)(1-\bar{z}),\ z-\bar{z},\ \frac{\bar{z}}{z},\ \frac{1-z}{1-\bar{z}} \right\}$$

Simplified alphabet (quicker for testing)

$$A_3^\star = \left\{ z\bar{z},\ (1-z)(1-\bar{z}),\ \frac{\bar{z}}{z},\ \frac{1-z}{1-\bar{z}} \right\}$$

Allow all possibilities up to weight 6

Weight-0: $1$

Weight-1: $G(a_1,x),\ \pi$

Weight-2: $G(a_1,a_2,x),\ \pi\times G(a_1,x),\ \zeta_2$

Weight-3: $G(a_1,a_2,a_3,x),\ \pi\times G(a_1,a_2,x),\ \zeta_2\times G(a_1,x),\ \zeta_3$

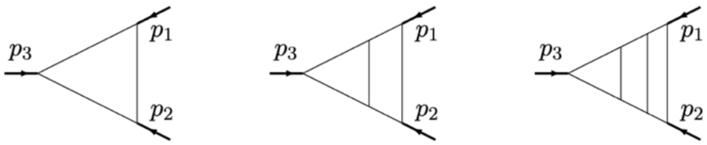
Weight-4: $G(a_1,a_2,a_3,a_4,x),\ \pi\times G(a_1,a_2,a_3,x),\ \zeta_2\times G(a_1,a_2,x),\ \pi^3\times G(a_1,x),$
$\zeta_3\times G(a_1,x),\ \zeta_4$

Weight-5: $G(a_1,a_2,a_3,a_4,a_5,x),\ \pi\times G(a_1,a_2,a_3,a_4,x),\ \zeta_2\times G(a_1,a_2,a_3,x),$
$\pi^3\times G(a_1,a_2,x),\ \zeta_3\times G(a_1,a_2,x),\ \zeta_4\times G(a_1,x),\ \zeta_5,\ \zeta_2\times\zeta_3$

Weight-6: $G(a_1,a_2,a_3,a_4,a_5,a_6,x),\ \pi\times G(a_1,a_2,a_3,a_4,a_5,x),\ \zeta_2\times G(a_1,a_2,a_3,a_4,x),$
$\pi^3\times G(a_1,a_2,a_3,x),\ \zeta_3\times G(a_1,a_2,a_3,x),\ \zeta_4\times G(a_1,a_2,x),\ \zeta_5\times G(a_1,x),$
$\zeta_2\zeta_3\times G(a_1,x),\ \pi^5\times G(a_1,x),\ \zeta_6,\ \zeta_3^2$

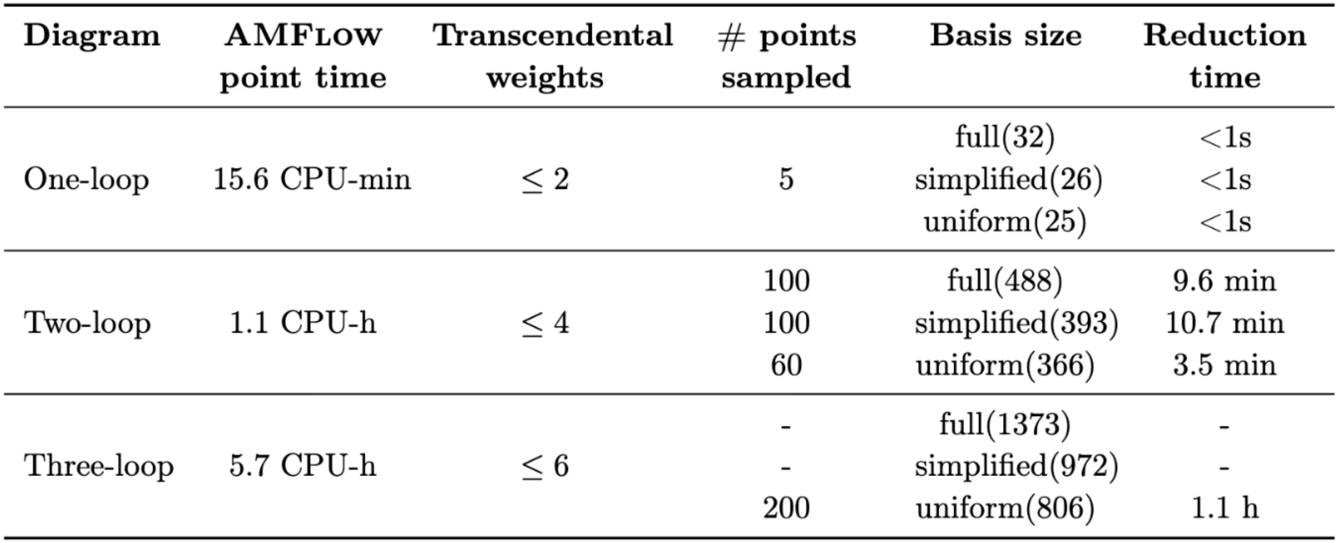
# Example 1: Triangles

Triangle ladder diagrams



Pick random points in (unphysical) Euclidean region $0 < z < \bar{z} < 1$

- Makes basis functions real
- Imaginary numbers are fine, just technically complicated python implemntation



| Diagram | AMFLOW point time | Transcendental weights | # points sampled | Basis size | Reduction time |
|---|---|---|---|---|---|
| One-loop | 15.6 CPU-min | $\leq 2$ | 5 | full(32) | <1s |
| | | | | simplified(26) | <1s |
| | | | | uniform(25) | <1s |
| Two-loop | 1.1 CPU-h | $\leq 4$ | 100 | full(488) | 9.6 min |
| | | | 100 | simplified(393) | 10.7 min |
| | | | 60 | uniform(366) | 3.5 min |
| Three-loop | 5.7 CPU-h | $\leq 6$ | - | full(1373) | - |
| | | | - | simplified(972) | - |
| | | | 200 | uniform(806) | 1.1 h |

3 digits

20 digits

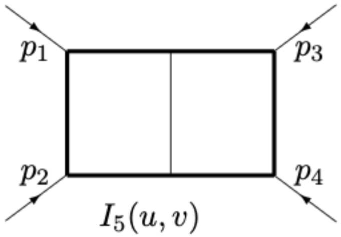
20 digits

Results with timing

- Rate limiting step is AMFlow (computation of full function)
- Could be sped up (additonal points much faster than first)

# Example 2: Double box

Loop



12 independent letters

$$\tilde{A} = \left\{u, v, 1+u, 1+v, u+v, 1+u+v, \frac{\beta_u - 1}{\beta_u + 1}, \frac{\beta_v - 1}{\beta_v + 1}, \frac{\beta_{uv} - 1}{\beta_{uv} + 1}, \frac{\beta_{uv} - \beta_u}{\beta_{uv} + \beta_u}, \frac{\beta_{uv} - \beta_v}{\beta_{uv} + \beta_v}, \frac{\beta_{uv} - \beta_u\beta_v}{\beta_{uv} + \beta_u\beta_v}\right\}$$

with $\beta_u = \sqrt{1+u}$, $\beta_v = \sqrt{1+v}$ and $\beta_{uv} = \sqrt{1+u+v}$.

- $12^4$ = 20,736 weight-4 symbols +(?) lower weight terms
- square root letters are hard to integrate analytically

method 1:
rationalize the square roots

$$u = \frac{(1-w^2)(1-z^2)}{(w-z)^2} \quad \text{and} \quad v = \frac{4wz}{(w-z)^2},$$

$$\tilde{A}_2 = \{w, z, 1 \pm w, 1 \pm z, w \pm z, 1 \pm wz, 1 \pm w \mp z + wz\} \,;$$

- Now symbols can be integrated analytically
  - Takes FiberSymbol hours to integrate
  - Result is hundreds or thousands of terms
- GiNaC  can get numbers out, but very slow

method 2:
numerically integate along a contour

- integrate first and last symbol analytically
- need to be careful with branch cuts
  - euclidean region requires some thought
- using integrable contributions helps a lot

# Example 2: Double box

Landau boostrap

Analytic regression

relatively easy

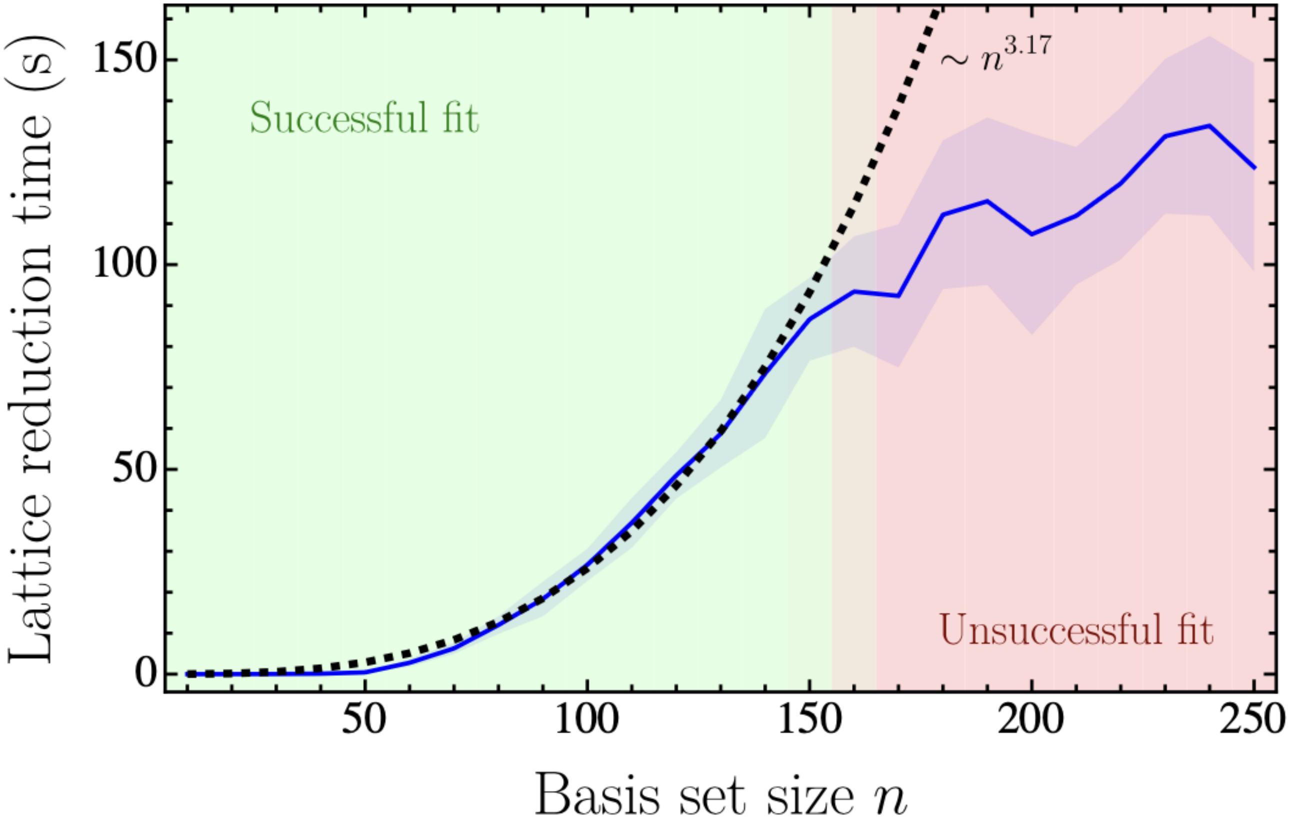
| Constraints | # Coeffs |
|---|---|
| All Symbols | 20736 |
| Integrability | 6993 |
| Galois symmetry | 861 |
| Physical branch cuts | 161 |
| Genealogical constraints | 28 |
| $\alpha$-positive thresholds | 6 |

✓ 200pts 17 digits, t ∽ 30s

✓ 200pts, 8 digits, t < 1s

✓ 200pts, 6 digits, t < 1s

# Double box: limitations



Should work with more functions

- With our compute, hard to succeed with more than n $\lesssim$ 200

# Choosing points

- Before we said you can get away with fewer digits if you use more points

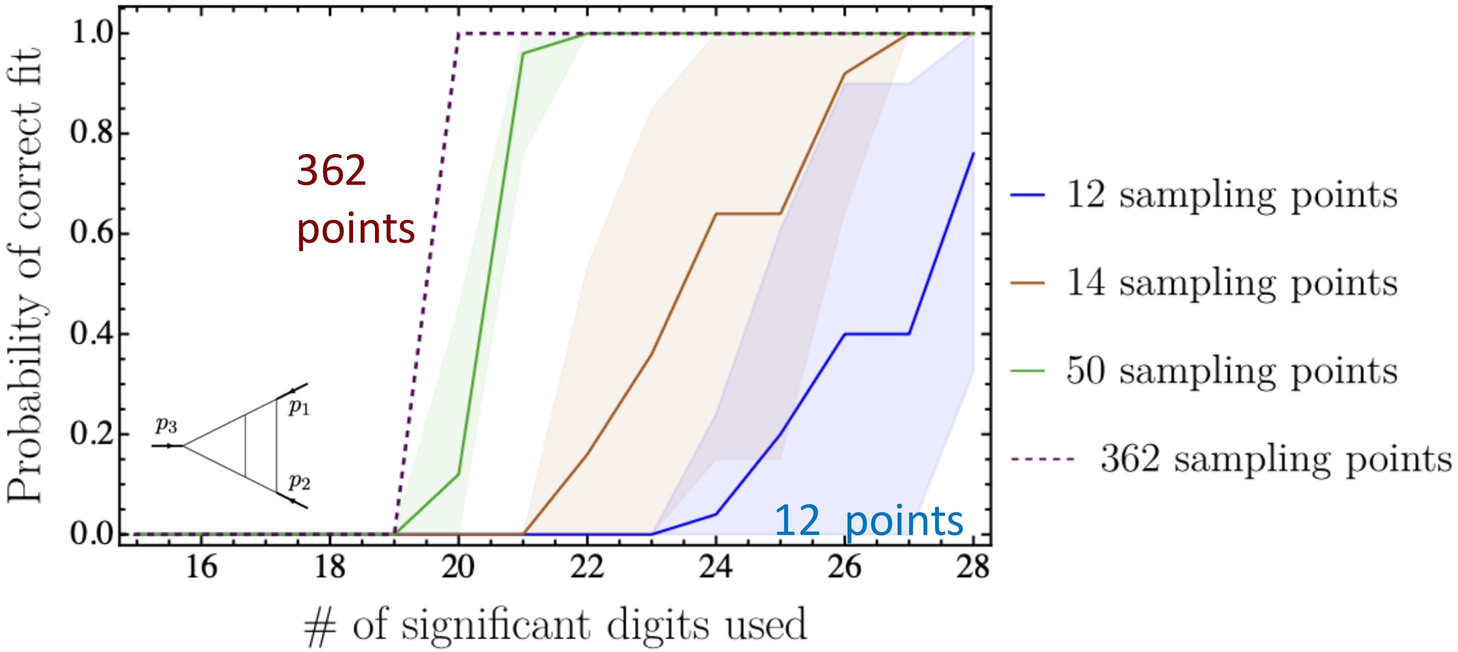
# basis functions

size of integers

digits of precision req’d

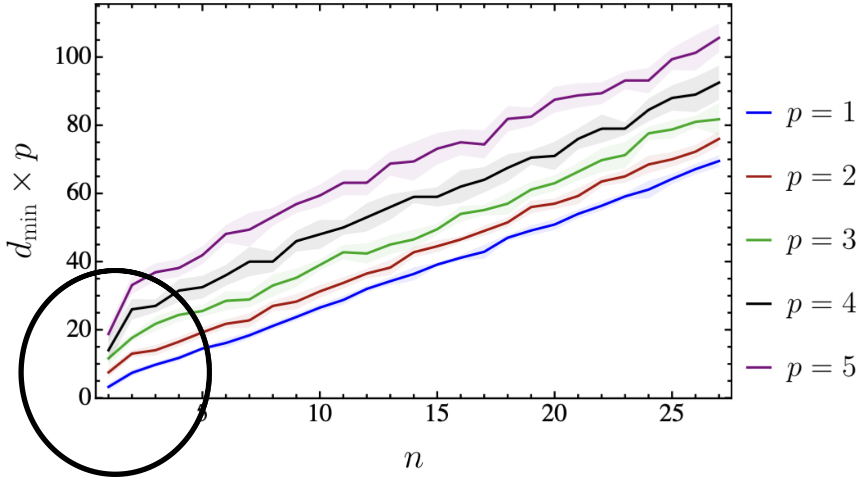
$$d \lesssim \frac{nR}{p}$$

number of points



- Can never succeed below some digit lower bound
- Why did scaling fail?

# Choosing points



offset near d=0

$$d_{\min} \approx R_{\text{eff}} \frac{n}{p} + d_0$$

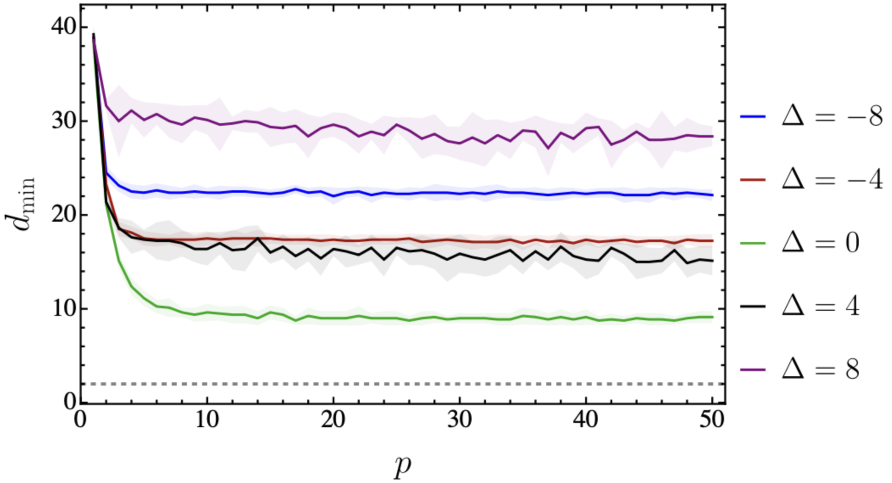
- will never work with 1 or 2 digits

$$f(p_j) = \{103.2, \underbrace{2.5, \quad 2.3}\}$$

unrecovarable information
loss if we trunctate to 2 digits

choose points in a range

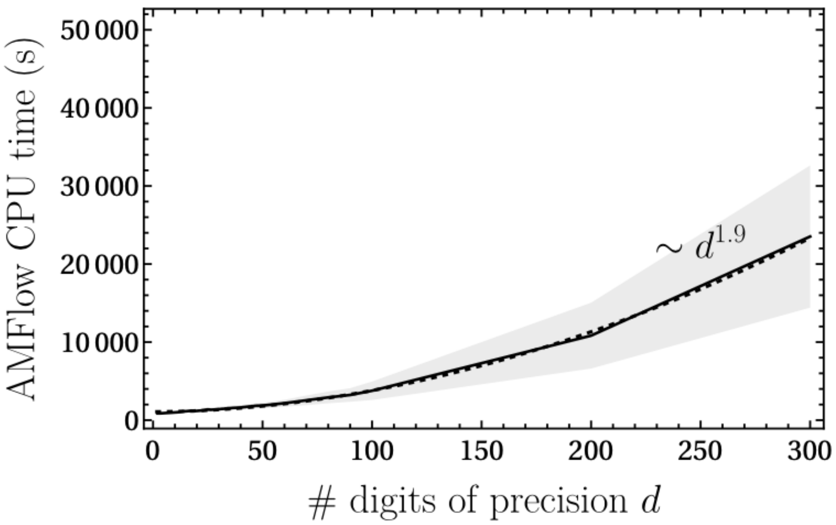
$$\tfrac{1}{2} - 10^{\Delta} \leq x \leq \tfrac{1}{2} + 10^{\Delta}$$



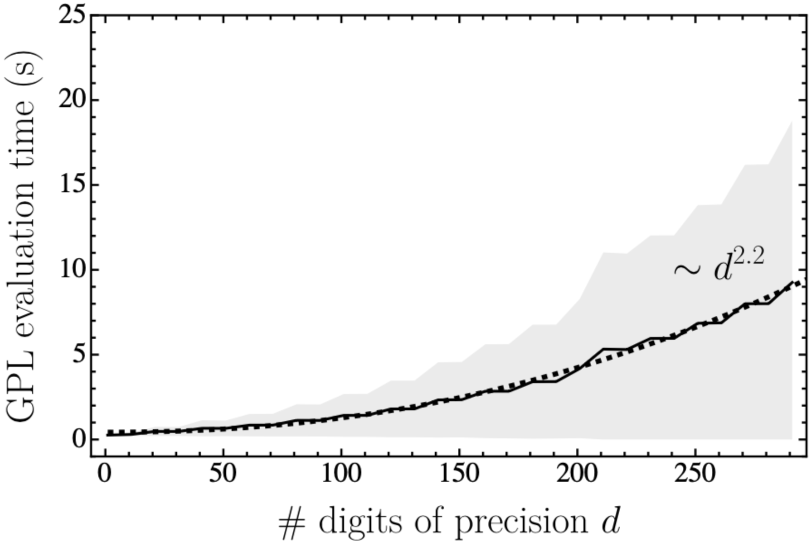
- want to choose points
  - not too close (lose information)
  - not too far (need more digits)
  - Need 10-20 digits at least

# Double box: timing

AMFlow + Kira (for IBP)



GPL evaluation (GINSH)



$$t(p,n)/\mathrm{ns} \approx \underbrace{10^9 \cdot p \cdot d_{\mathrm{min}}^2(n)}_{\textsc{AMFlow}} + \underbrace{10^4 \cdot n \cdot p \cdot d_{\mathrm{min}}^2(n)}_{\textsc{Ginsh}} + \underbrace{p \cdot n^4}_{\text{fitting}},$$

- Trading digits for points makes scaling go from quadradic to linear!

# Application to EECs

Toward the Analytic Bootstrap of Energy Correlators

Jianyu Gong[a,b] Andrzej Pokraka[c] Kai Yan[a,b] Xiaoyuan Zhang[d,e]

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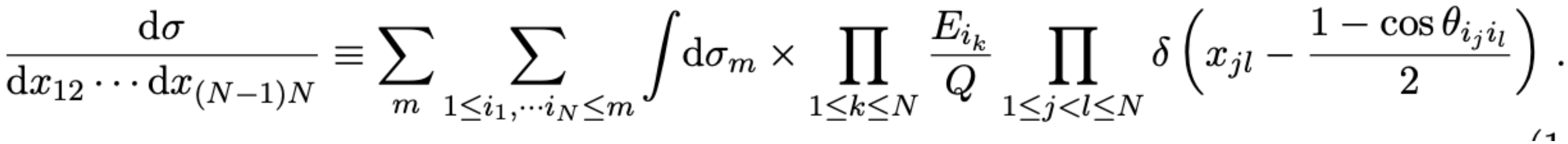
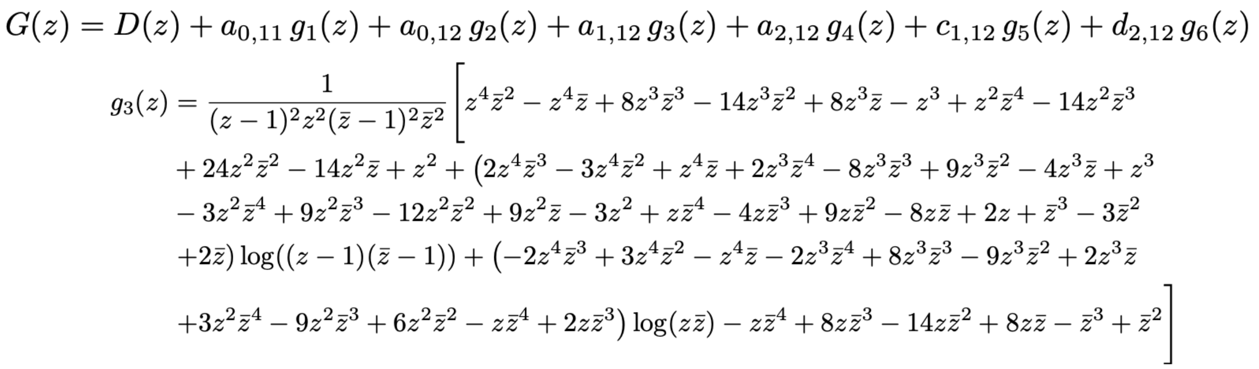
Consider some types of energy-energy correlators (unequal energy weights)



$$\frac{\mathrm{d}\sigma}{\mathrm{d}x_{12}\cdots\mathrm{d}x_{(N-1)N}} \equiv \sum_m \sum_{1\leq i_1,\cdots i_N\leq m} \int \mathrm{d}\sigma_m \times \prod_{1\leq k\leq N} \frac{E_{i_k}}{Q} \prod_{1\leq j<l\leq N} \delta\left(x_{jl} - \frac{1-\cos\theta_{i_j i_l}}{2}\right).$$

Uses of lattice reducion

- Find linear dependence among basis functions



$$G(z) = D(z) + a_{0,11}\, g_1(z) + a_{0,12}\, g_2(z) + a_{1,12}\, g_3(z) + a_{2,12}\, g_4(z) + c_{1,12}\, g_5(z) + d_{2,12}\, g_6(z)$$

$$\begin{aligned} g_3(z) = \frac{1}{(z-1)^2 z^2 (\bar{z}-1)^2 \bar{z}^2} \Big[ & z^4\bar{z}^2 - z^4\bar{z} + 8z^3\bar{z}^3 - 14z^3\bar{z}^2 + 8z^3\bar{z} - z^3 + z^2\bar{z}^4 - 14z^2\bar{z}^3 \\ & + 24z^2\bar{z}^2 - 14z^2\bar{z} + z^2 + \big(2z^4\bar{z}^3 - 3z^4\bar{z}^2 + z^4\bar{z} + 2z^3\bar{z}^4 - 8z^3\bar{z}^3 + 9z^3\bar{z}^2 - 4z^3\bar{z} + z^3 \\ & - 3z^2\bar{z}^4 + 9z^2\bar{z}^3 - 12z^2\bar{z}^2 + 9z^2\bar{z} - 3z^2 + z\bar{z}^4 - 4z\bar{z}^3 + 9z\bar{z}^2 - 8z\bar{z} + 2z + \bar{z}^3 - 3\bar{z}^2 \\ & + 2\bar{z}\big) \log((z-1)(\bar{z}-1)) + \big(-2z^4\bar{z}^3 + 3z^4\bar{z}^2 - z^4\bar{z} - 2z^3\bar{z}^4 + 8z^3\bar{z}^3 - 9z^3\bar{z}^2 + 2z^3\bar{z} \\ & + 3z^2\bar{z}^4 - 9z^2\bar{z}^3 + 6z^2\bar{z}^2 - z\bar{z}^4 + 2z\bar{z}^3\big) \log(z\bar{z}) - z\bar{z}^4 + 8z\bar{z}^3 - 14z\bar{z}^2 + 8z\bar{z} - \bar{z}^3 + \bar{z}^2 \Big] \end{aligned}$$

- Treat $\{g_i\}$ as transcendental basis and $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}$ as coefficients to determine
- (1). Run lattice reduction among $\{g_i(z)\}$ to get a linear-independent basis $\{\tilde{g}_i(z)\}$
- (2). Run lattice reduction among both $G(z)$ and $\{\tilde{g}_i(z)\}$

In this case, 2 numerical points and 13 digits fix all six parameters

# Pros and cons: Landau bootstrap

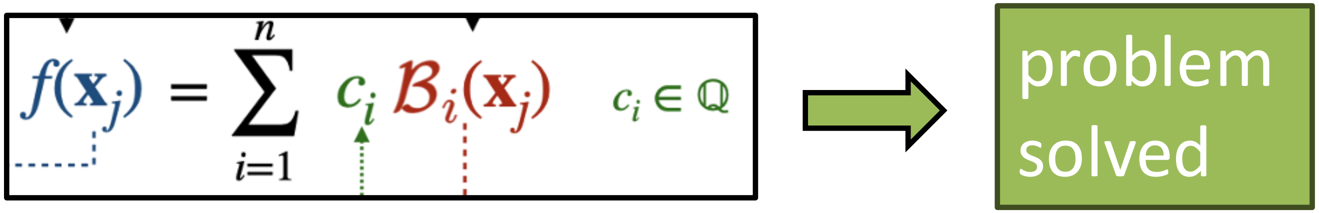
- Can eliminate large swaths of symbols with physical constraints
- Don’t need to do integrals
- Leads to new deep understanding in what amplitdues are

- Requires subtle understanding of singularities
    - Analytic structure of amplitudes
    - Branch points, euclidean regions
    - Algebraic geometry
- Rational prefactors not fixed by singularities alone
- Often still requires some integration at the end

# Pros and cons: Lattice reduction

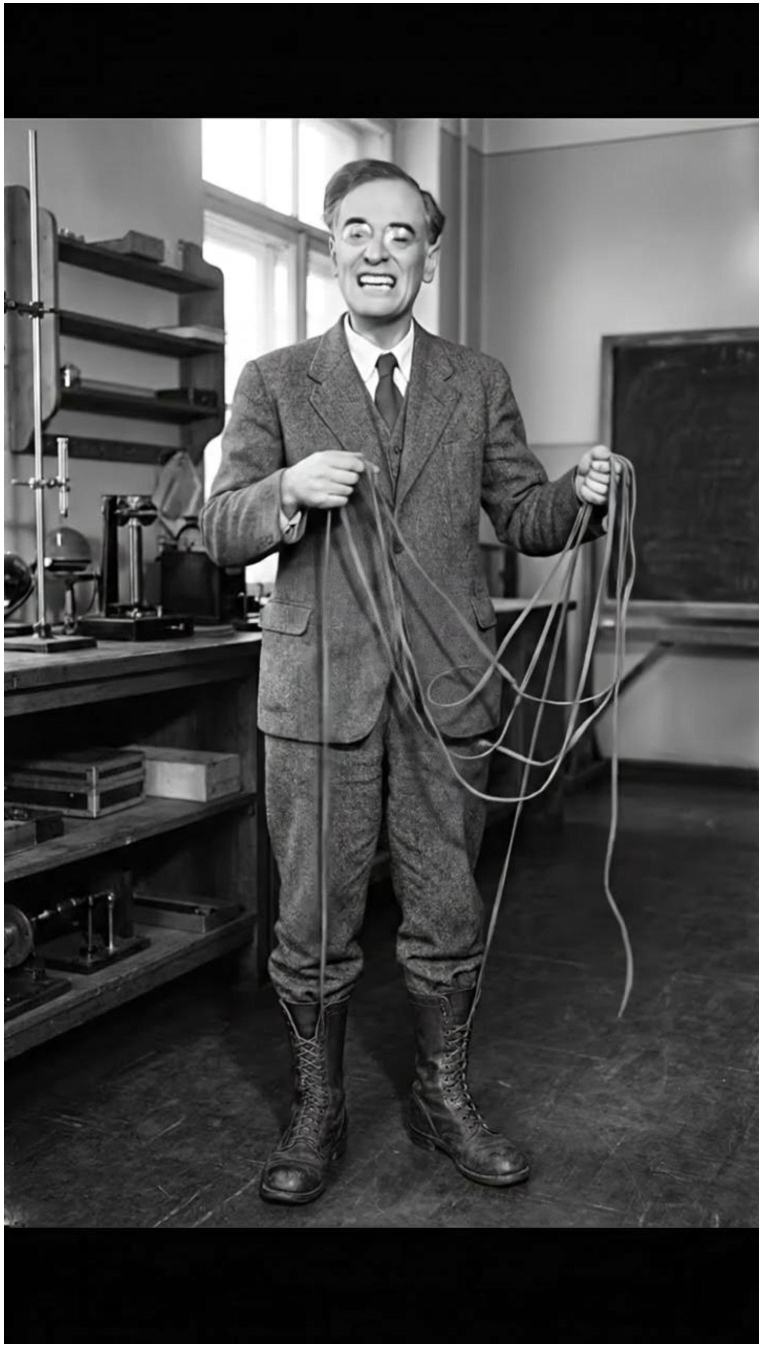
- Easy to automate
- Can work for any functions
  - not just polylogs with symbols
  - elliptic polylarithms? no problem!
  - cross sections, EECs, etc.
- Can find linear dependencies
- Can trade digits of accuracy for points
  - Scaling t~n$^2$ or worse to t~n

- Becomes computationally challenging above n ~ 200
  - AMFlow scales like (# digits)$^2$
  - lattice reduction scales like (# constants)$^4$
- Minimum number of digits needed (~ 5 or 6)



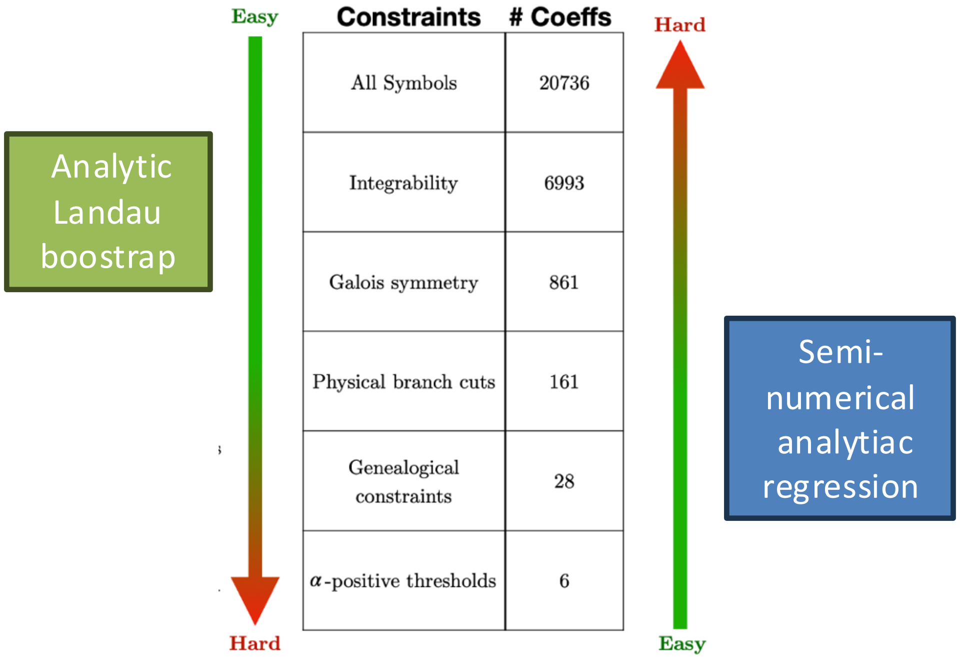
$$f(\mathbf{x}_j) = \sum_{i=1}^{n} c_i \, \mathcal{B}_i(\mathbf{x}_j) \qquad c_i \in \mathbb{Q}$$

problem solved



# Conclusions

- Bootrapping Feynman integrals is now possible in a semi-automated way
- Analytic and numerical methods are complementary



Analytic Landau boostrap

Easy

Hard

| Constraints | # Coeffs |
|---|---|
| All Symbols | 20736 |
| Integrability | 6993 |
| Galois symmetry | 861 |
| Physical branch cuts | 161 |
| Genealogical constraints | 28 |
| $\alpha$-positive thresholds | 6 |

Hard

Easy

Semi-numerical analytiac regression

- Semi-numerical analytic regression could be widely valuable for many tasks