What is the Fate of the Universe?

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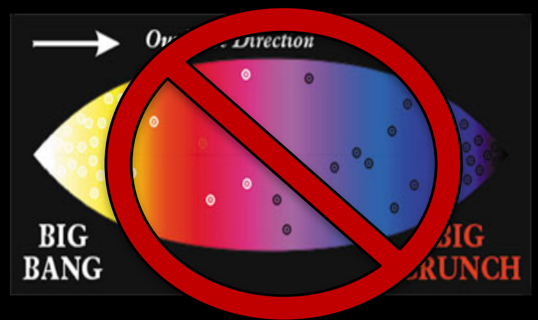
Based on PRD [in review] (arXiv:1707.08124)
PRD (95) 085011 (arXiv:1604.06090)
PRL 117.231601 (arXiv:1602.01102)
PRD (91) 016009 (arXiv:1408.0287)
PRL 113.241801 (arXiv:1408.0292)
with Anders Andreassen, David Farhi and William Frost

What is the fate of the universe?

1. Static universe

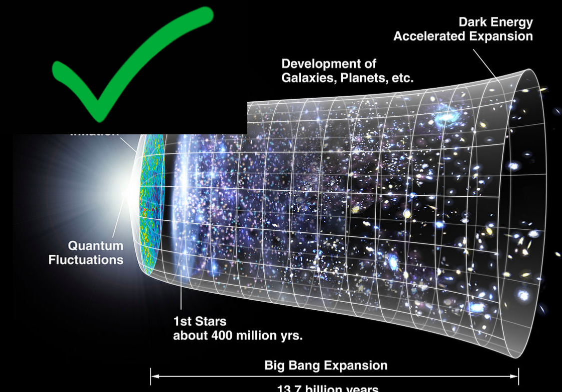
Einstein 1917:
Dark energy tuned against matter

2. Big Crunch

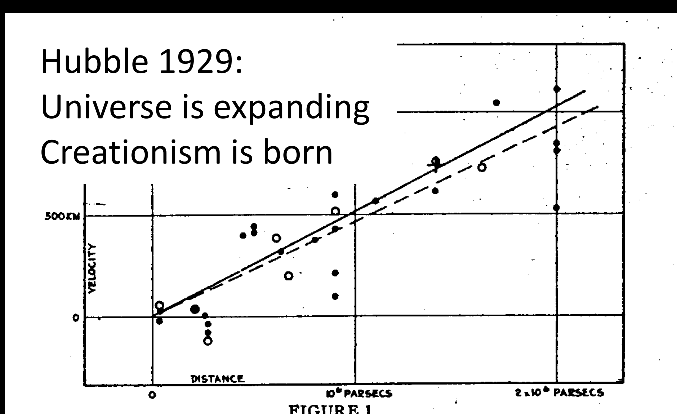


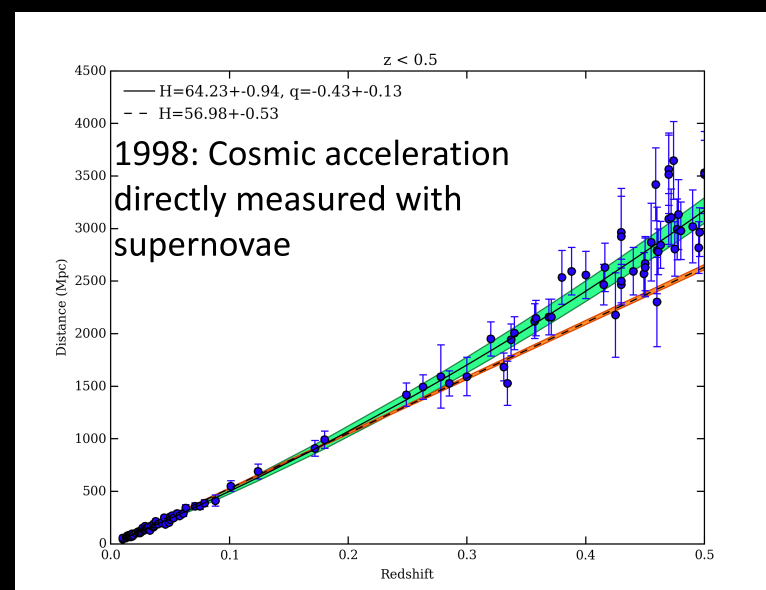
$\Lambda \leq 0$

3. Cold and Empty Future

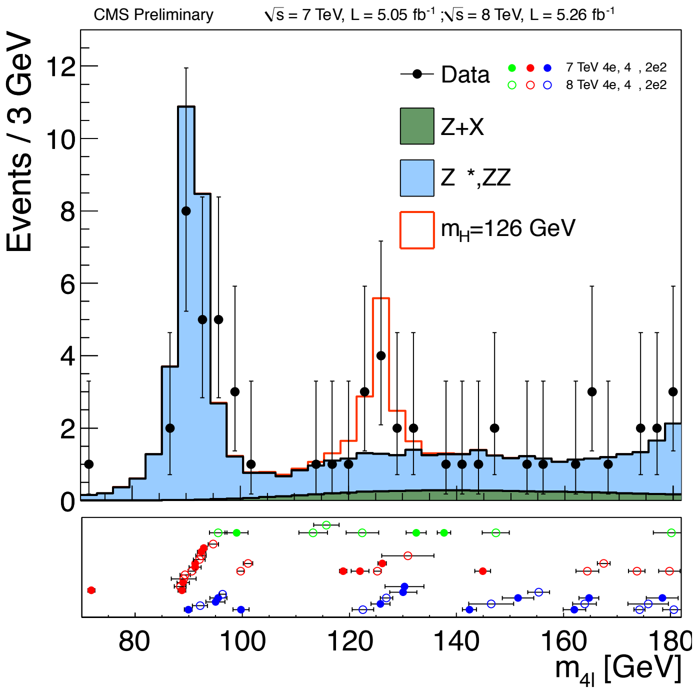


$\Lambda > 0$





July 4, 2012: Higgs boson discovered!



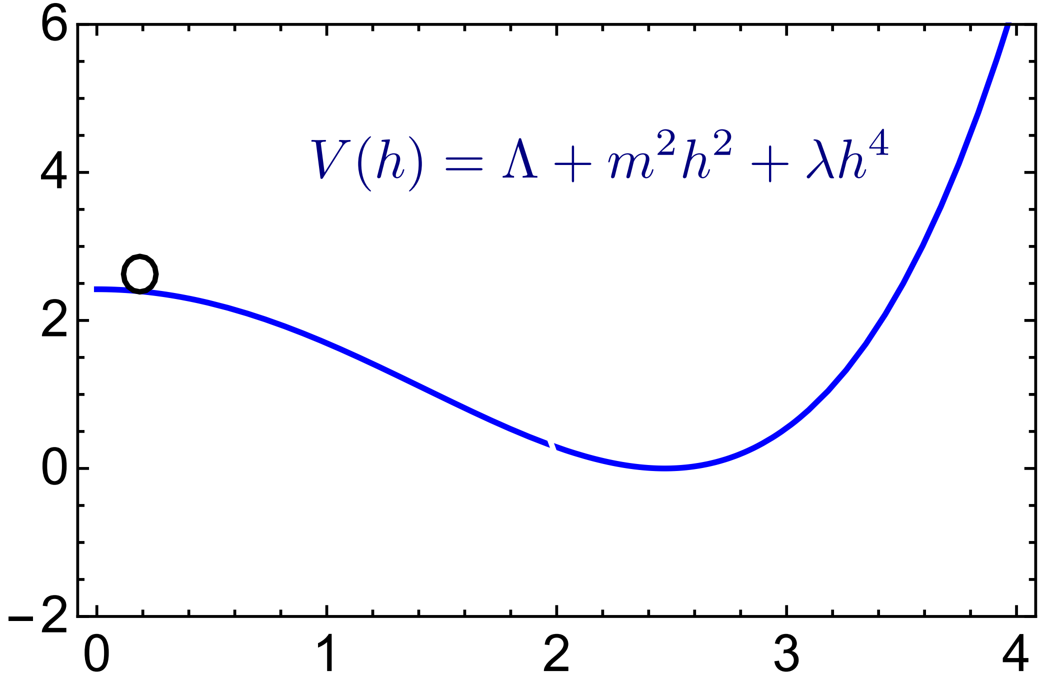
What did we learn?

What is the Higgs field?

- The Higgs field h(x) has a constant nonzero value everywhere

$$\langle h \rangle = v = 246 \text{ GeV}$$

- Excitations of the Higgs field are Higgs bosons
- How hard it is to excite the Higgs field depends on its potential



What do we know about this Higgs potential?

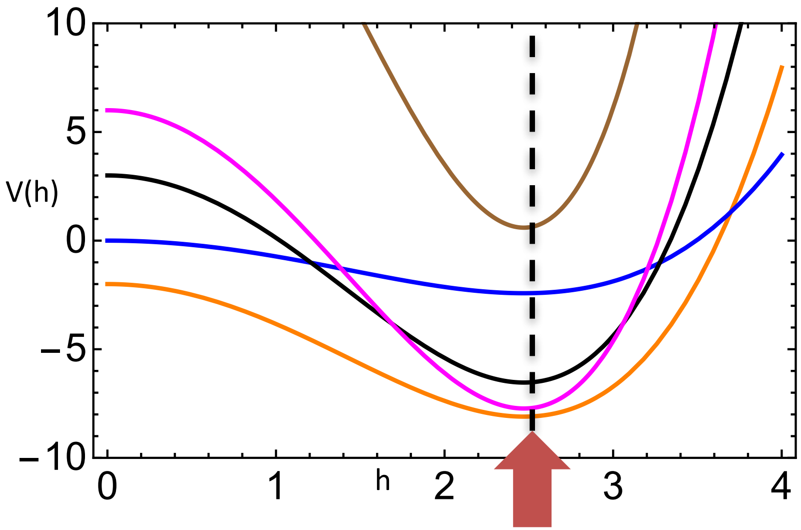
Renormalizability of the Standard Model

Classical potential is quartic (4th order)

$$V(h) = \Lambda + m^2 h^2 + \lambda h^4$$

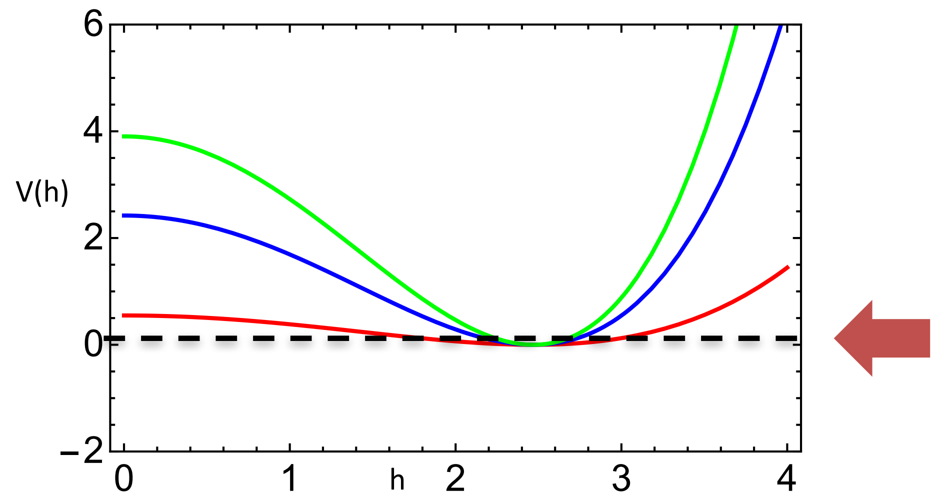
- 3 free parameters (Λ, m λ)
  - Must be measured from data

Higgs potential



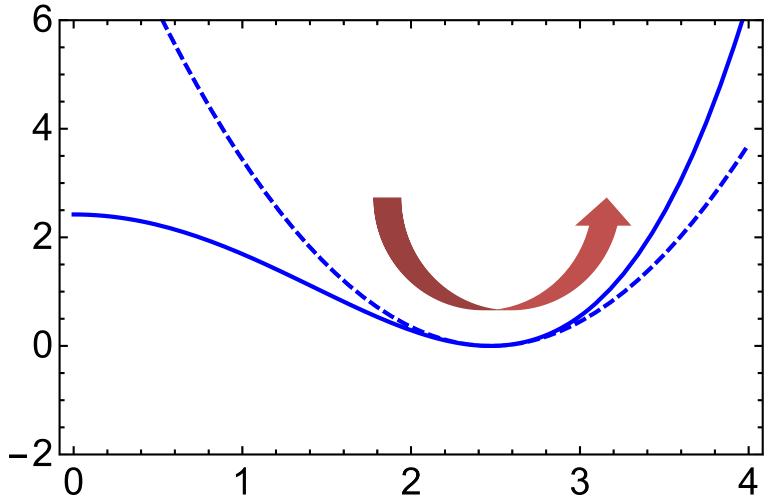
1933: Fermi theory of radioactive decay

$$\langle h \rangle = v = G_F^{-1/2} = 246\ \mathrm{GeV}$$



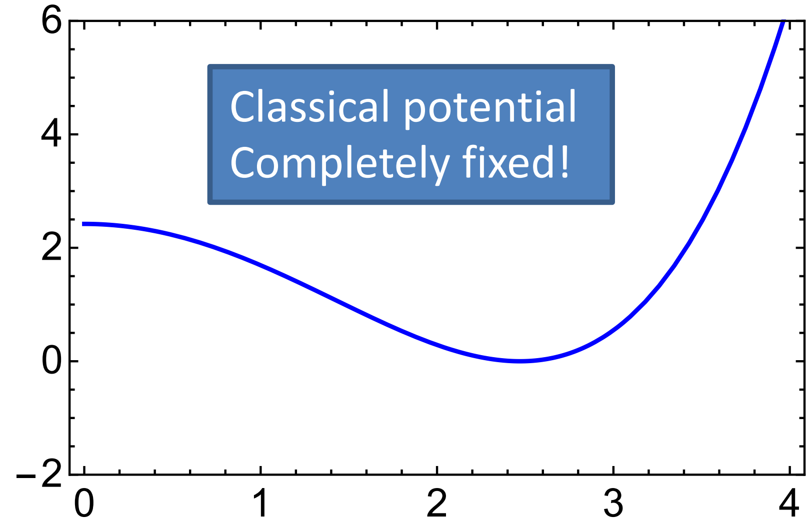
1998: zero-point energy (dark energy) measured

$$V(v) = (10^{-3}\ \mathrm{eV})^4$$



2012: curvature at minimum measured

$$V''(v) = m_h^2 = (126\ \mathrm{GeV})^2$$



Renormalizability of the Standard Model

Classical potential is quartic (4th order)

$$V(h) = \Lambda + \frac{1}{2}m^2(h-v)^2$$

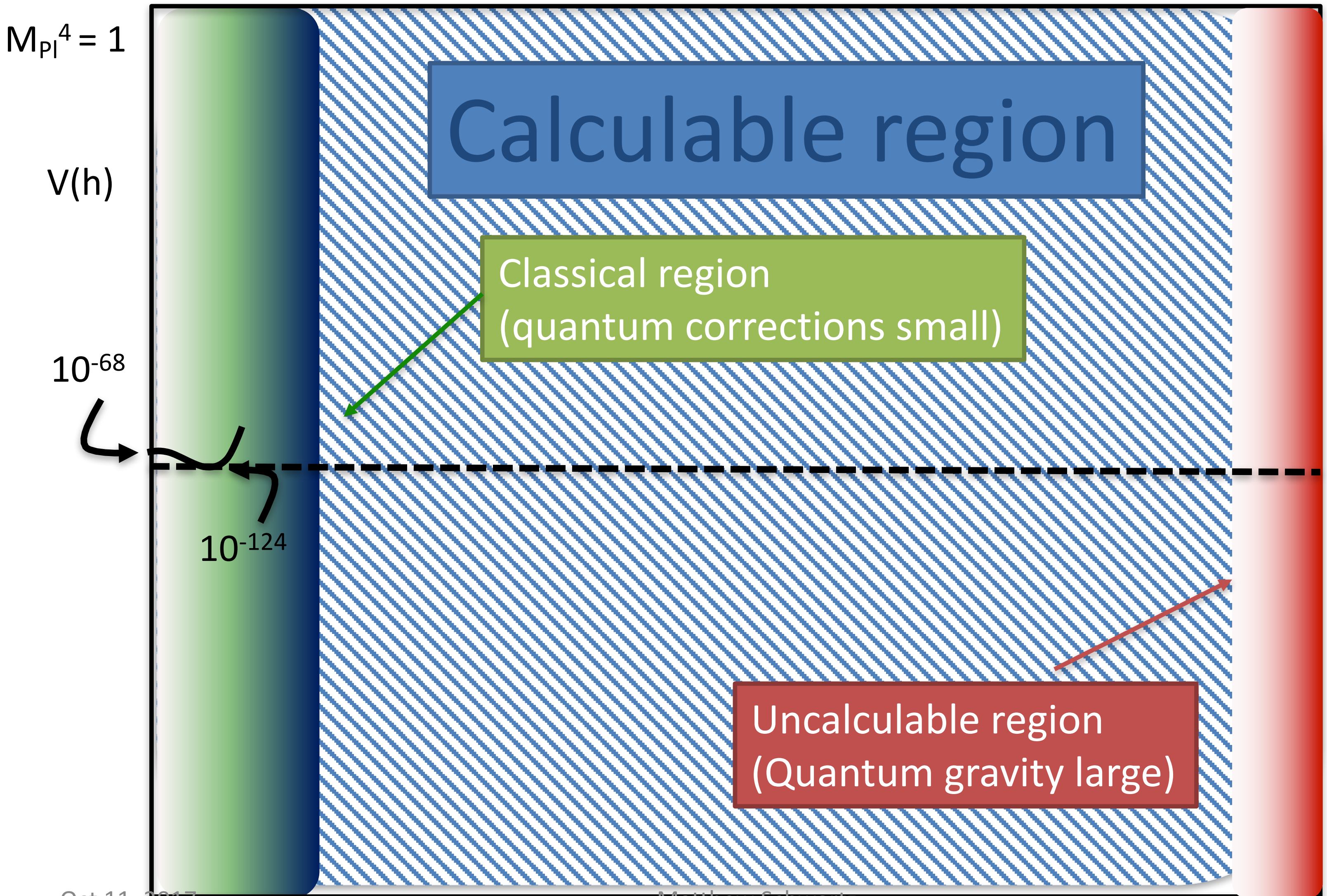
- 3 free parameters (Λ, m λ)
  - Must be measured from data ✓

quantum-corrected or Effective Potential

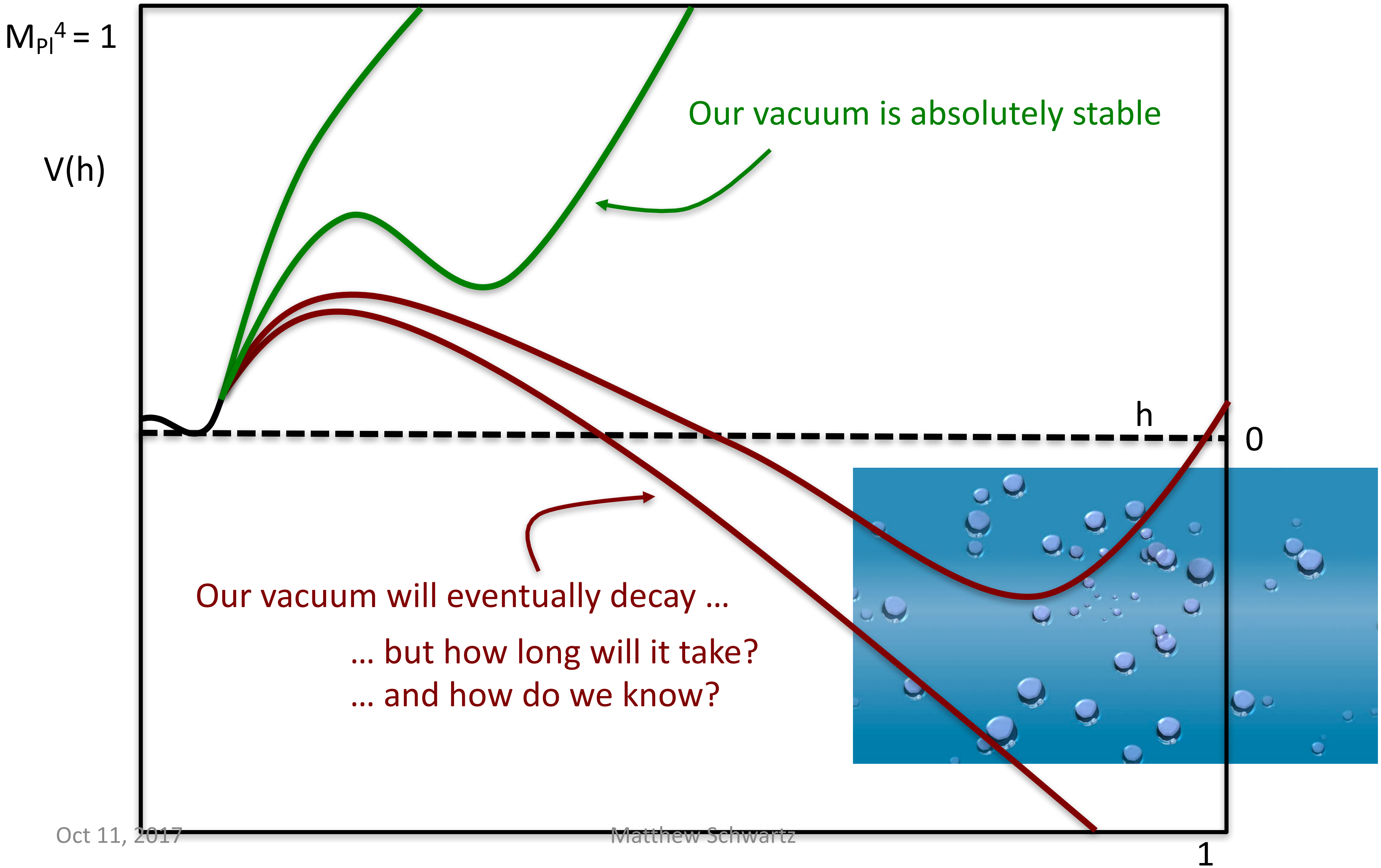
$$V(h) = \Lambda + \frac{1}{2}m^2(h-v)^2 + \frac{\lambda^2}{16\pi^2}h^4 \ln\frac{h}{v} + \cdots$$

- Quantum corrections are **calculable**!
- Small corrections for $h \sim v$
- Can be large for $h \gg v$
- Limit on calculability is $h \lesssim M_{\rm Pl} = 10^{19}\ {\rm GeV}$

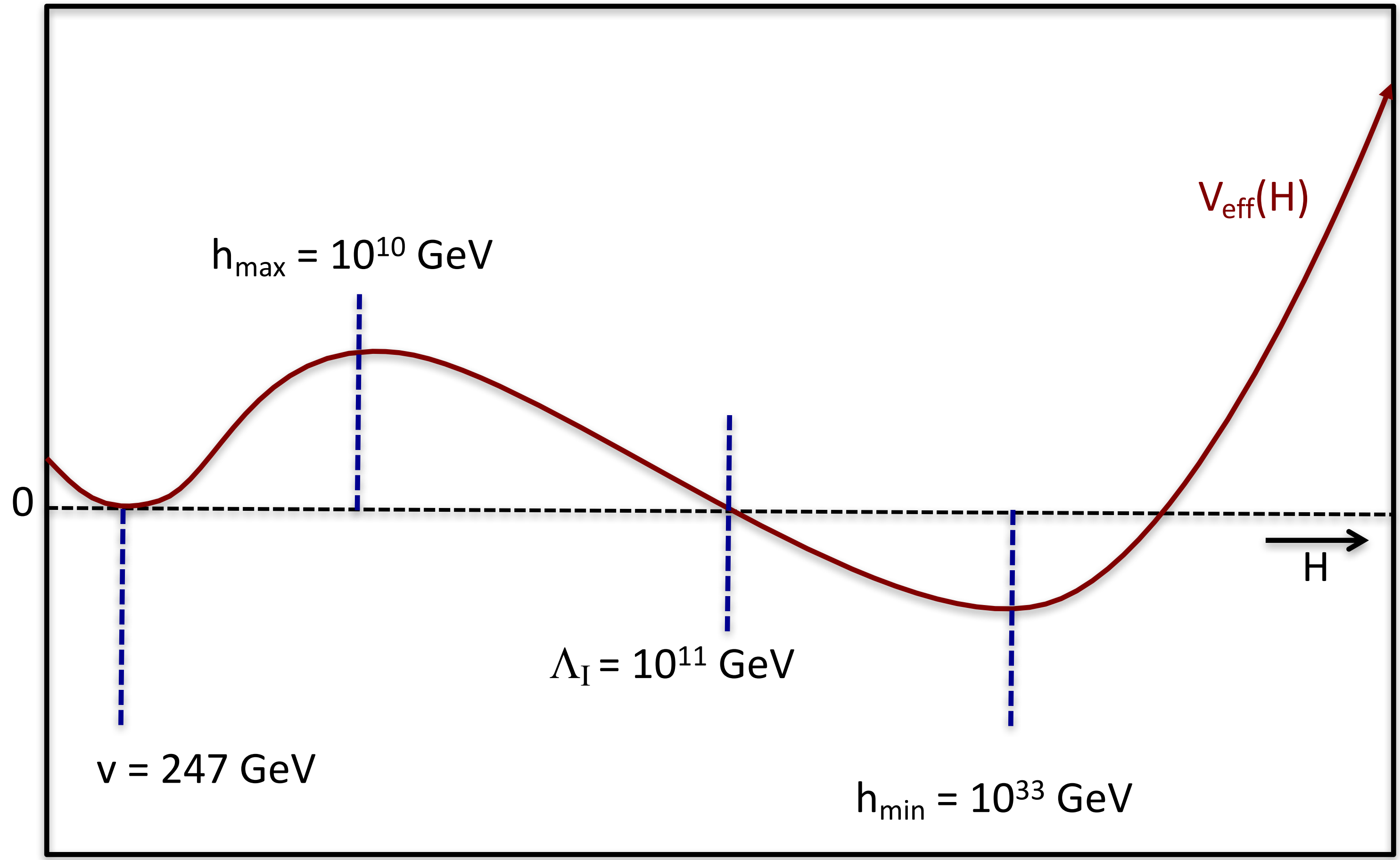
Natural units $M_{pl} = 1$



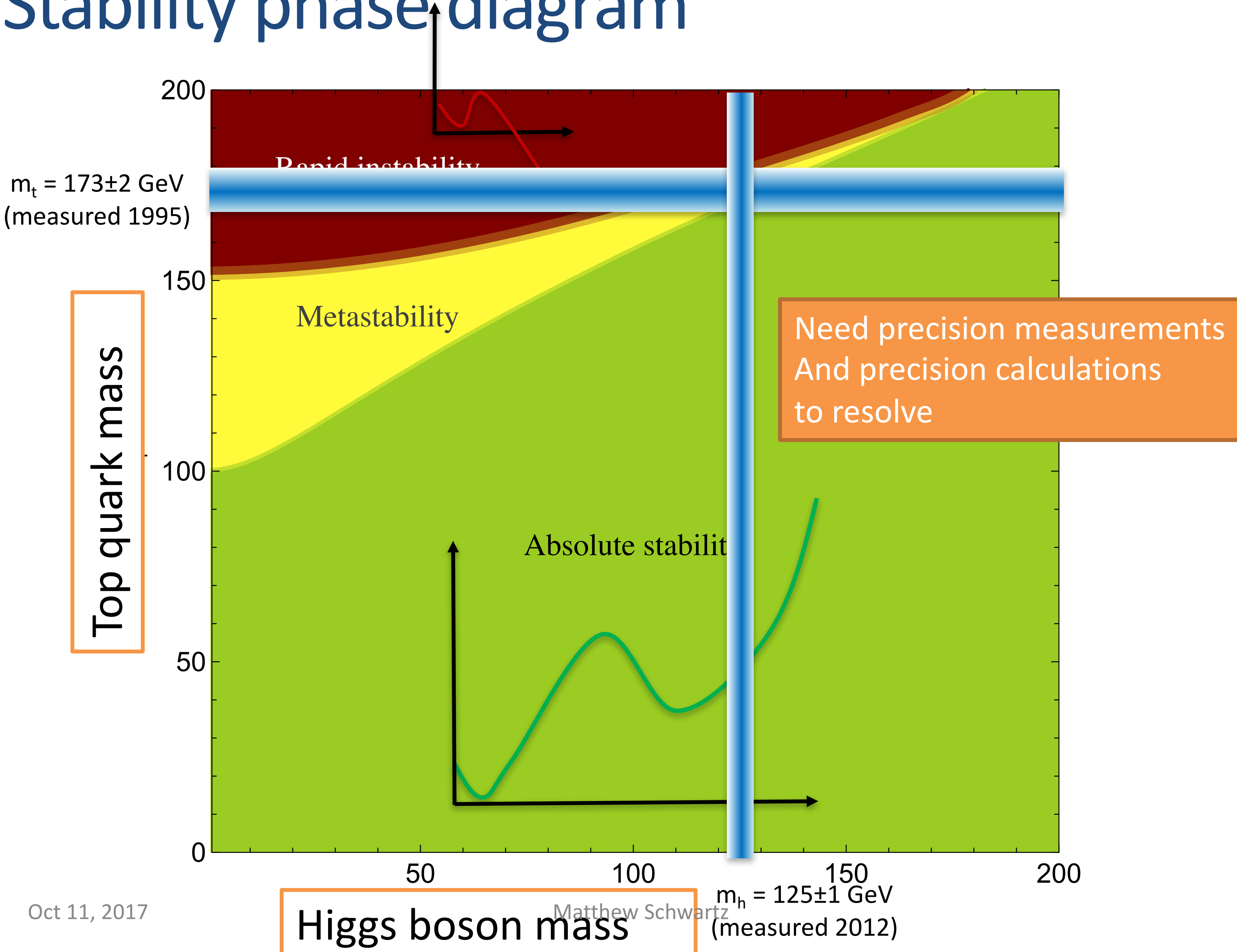
What could happen?



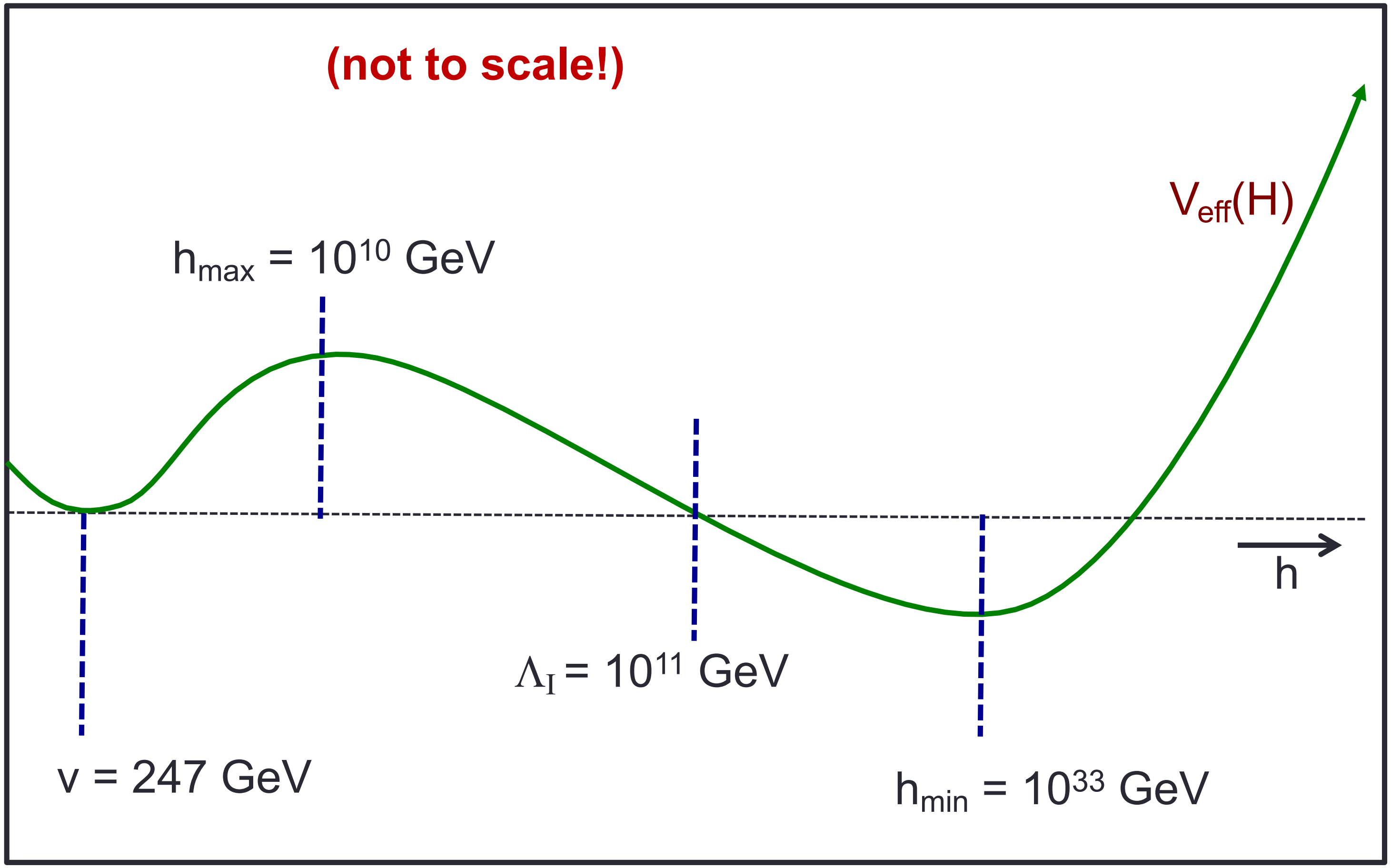
Standard Model Effective Potential



Stability phase diagram



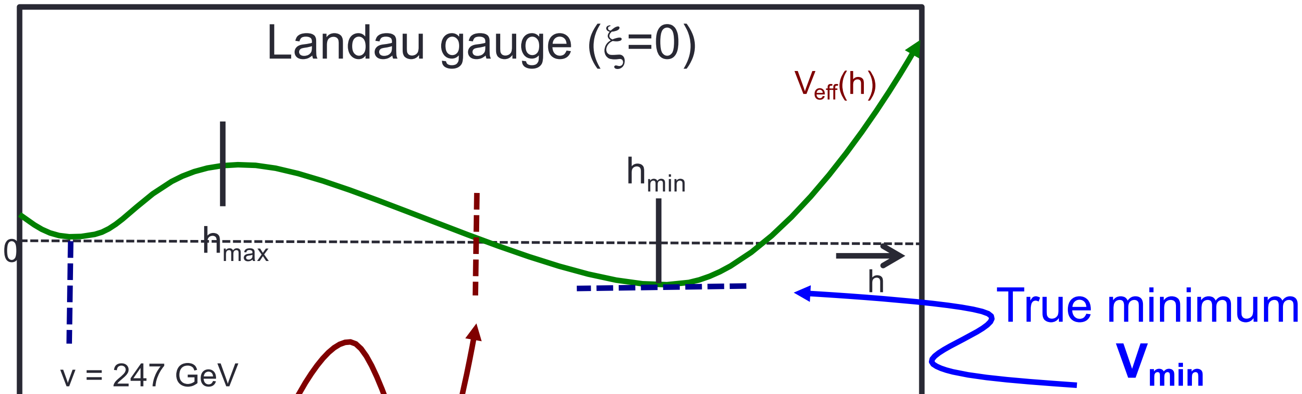
Standard Model Effective Potential



Are these scales physical?

Is the stability Planck sensitive?

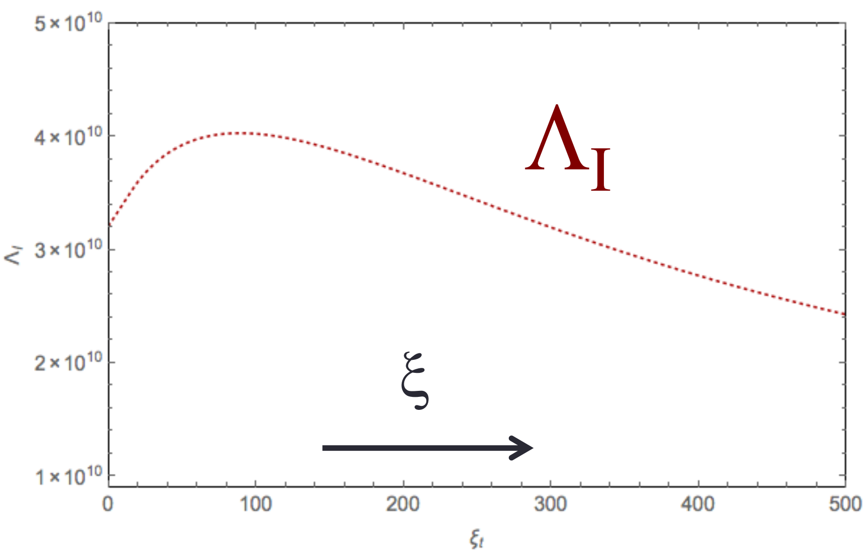
Problem 1: gauge dependence

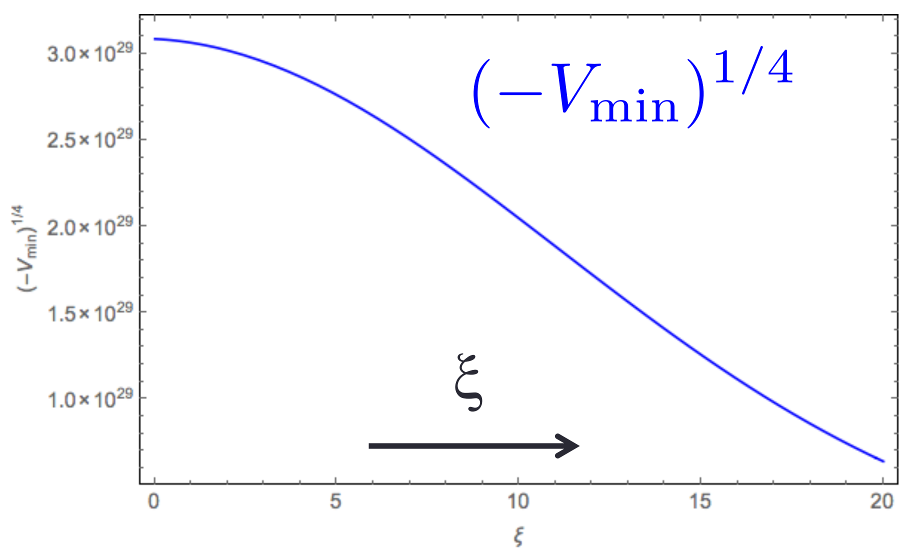


$V_{min} > 0$ → Absolute stability

Instability scale Λ_I
= value of h where V(h) = 0

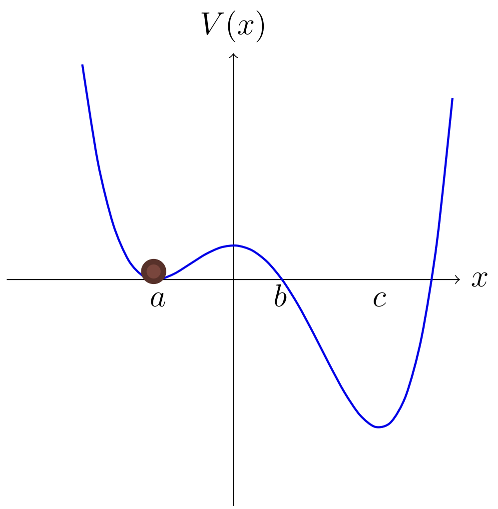
- Indicates sensitivity to new physics





- h_{min} also gauge dependent
- h_{max} also gauge dependent
- …

Problem 2: Γ = 0



$$Z \equiv \langle a|e^{-H\mathcal{T}}|a\rangle = \int_{x(0)=a}^{x(\mathcal{T})=a} \mathcal{D}x e^{-S_E[x]}$$

$$Z = \sum_E e^{-ET}|\psi_E(a)|^2$$

Isolate ground state energy from late times

$$E_0 = -\lim_{\mathcal{T}\to\infty}\frac{1}{\mathcal{T}}\ln Z$$

Decay rate is the imaginary part of the energy

$$\frac{\Gamma}{2} = \text{Im}\lim_{\mathcal{T}\to\infty}\frac{1}{\mathcal{T}}\ln Z$$

Clearly this is not exactly what is meant

- **Z is real**
- True ground state at $E_c = V(c)$ has nothing to do with the false vacuum

How do we get an imaginary part?

Problem 3: $\Gamma = \infty$

Assume the “standard” formula:

$$\frac{\Gamma}{2} = \mathrm{Im} \lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \ln \int \mathcal{D}\phi e^{-S[\phi_b]-\frac{1}{2}S''[\phi_b]\phi^2}$$

ϕ_b = “bounce” = solution to the Euclidean equations of motion $\Box\phi - V'(\phi_b) = 0$

Massive case (Higgs potential)

$$V'(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 \Longrightarrow$$ No solutions

Massless case (scaleless potential)

$$V'(\phi) = \frac{1}{4}\lambda\phi^4 \Longrightarrow$$ Too many solutions $$\phi_b^{R,x_0^\mu}(x) = \sqrt{\frac{8}{-\lambda}}\frac{R}{R^2 + (x+x_0)^2}$$

$$\frac{\Gamma}{V} \sim \# \Longleftarrow \Gamma \sim \lim_{T\to\infty} T\cdot V\# \Longleftarrow$$ x_0 (translations)

- rate per unit volume
- bubbles of true vacuum can form anywhere
- not a problem

$$\Gamma \sim \int_0^\infty dR = \infty \Longleftarrow$$ R (rescalings)

- bubbles of any size can form
- rate is infinite!

Summary

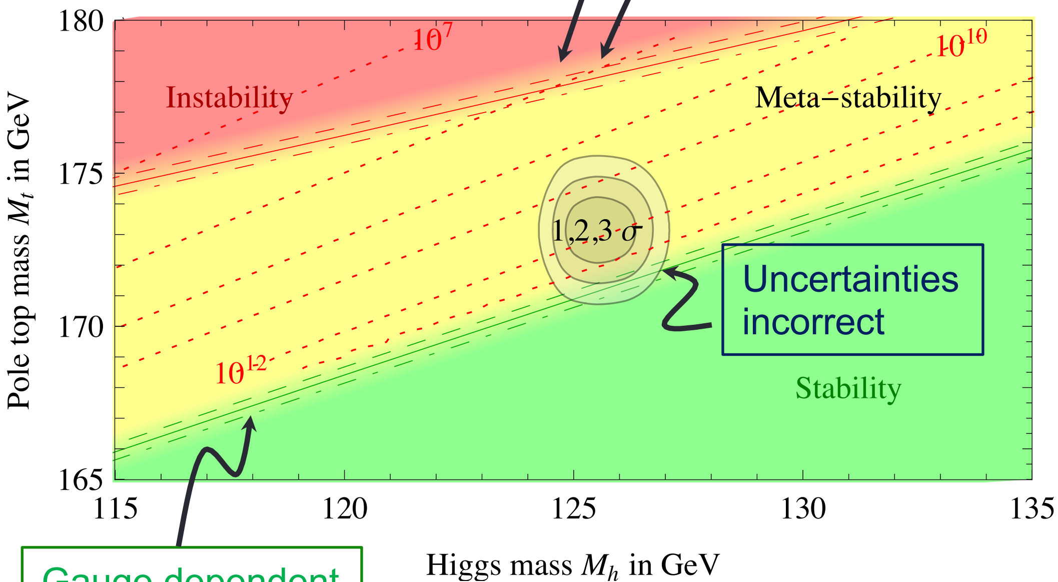
Andreassen, Farhi, Frost, MDS (various)

Previous work

DeGrassi et al 2012 (arXiv:1205.6497)

Unnecessary IR cutoffs used

Numerical approximations used



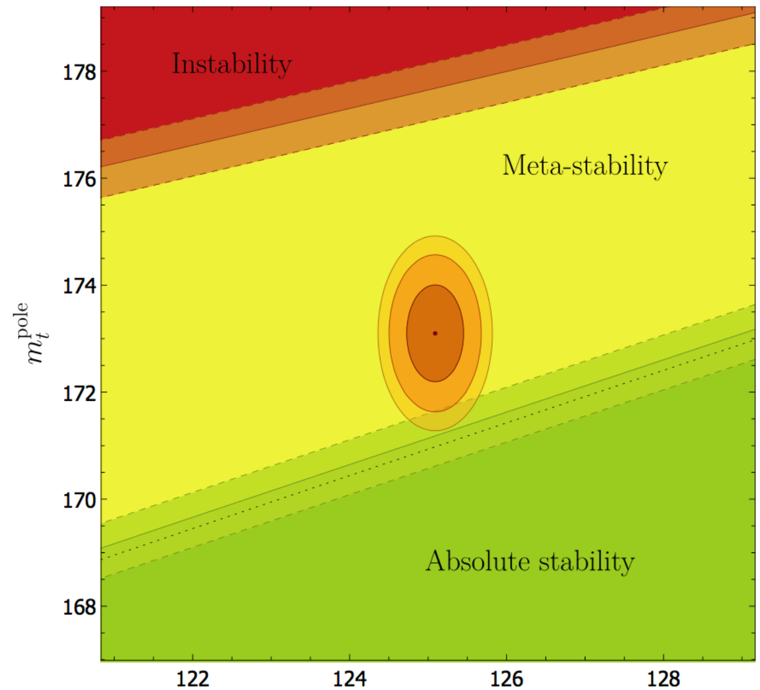
Uncertainties incorrect

Gauge dependent

- Improper use of renormalization group
- Based on heuristic Coleman-Callan formula

Our work

- No IR cutoffs
- Completely analytic rate formula



- Gauge invariant formula used
- Gauge invariance checked explicilty
- Uncertainties handled carefully
- Decay rate derived rigorously in QFT

COMPUTING FUNCTIONAL DETERMINANTS

How do we calculate a decay rate?

NLO formula: $$\frac{\Gamma}{2} = \mathrm{Im} \lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \ln \int \mathcal{D}\phi e^{-S[\phi_b]-\frac{1}{2}S''[\phi_b]\phi^2}$$

Step 1: Find the bounce

$\phi_b =$ “bounce” = solution to the Euclidean equations of motion

$$\Box\phi - V'(\phi_b) = 0$$

$$V(\phi) = \frac{1}{4}\lambda\phi^4 \quad \Longrightarrow \quad \phi_b^{R,x_0^\mu}(x) = \sqrt{\frac{8}{-\lambda}}\frac{R}{R^2+(x+x_0)^2}$$

5 parameter family of bounce solutions

x_0 (translations) $\Longrightarrow \quad \Gamma \sim \lim_{T\to\infty} T\cdot V\# \quad \Longrightarrow \quad \frac{\Gamma}{V} \sim \#$

- rate per unit volume
- bubbles of true vacuum can form anywhere

R (rescalings) $\Longrightarrow \quad \Gamma \sim \int_0^\infty dR = \infty$

- Bubbles of any size can form

Warm up: 1D scalar theory

Step 2: Reduce to math problem

$$V(\phi) = \frac{1}{4}\lambda\phi^4 \Longrightarrow \phi_b^{R,x_0^\mu}(x) = \sqrt{\frac{8}{-\lambda}}\frac{R}{R^2+(x+x_0)^2}$$

Need to compute

$$Z = \int \mathcal{D}\phi e^{-S[\phi_b]-\frac{1}{2}\phi S''[\phi_b]\phi} = \frac{1}{\sqrt{\det S''[\phi_b]}}e^{-S[\phi_b]}$$

Action is finite:

$$S[\phi_b] = \int d^4x\left[\frac{1}{2}(\partial_\mu\phi_b)^2 + \frac{1}{4}\lambda\phi_b^4\right] = -\frac{8\pi^2}{3\lambda} > 0$$

$$\det S''[\phi_b] = \prod_j \lambda_j$$

Compute determinant by multiplying eigenvalues

$$S''[\phi_b]\phi_j = (-\Box + 3\lambda\phi_b^2)\phi_j = \lambda_j\phi_j$$

- How do you find the eigenvalues???
- How do you multiply them together (infinite product)?

A. Just do it ✓
B. Use "elementary Fredholm theory" [Coleman Erice lectures, p. 340]
   - a.k.a. Gelfand-Yaglom method

Step 3: Solve math problem

A. Explicit eigenvalues

Want to solve

$$(-\Box + 3\lambda\phi_b^2)\phi_j = \lambda_j \phi_j$$

Choose one R and one center $\phi_b(x) = \sqrt{\frac{8}{-\lambda}}\frac{R}{R^2+r^2}$

5 modes with λ_j= 0:

4 translation modes $\phi_T = \partial_\mu \phi_b$ → $S''[\phi_b]\partial_\mu\phi_b = \partial_\mu(S'[\phi_b]) = 0$

- Finite Jacobian going to "collective coordinates"
- Integral over x0 gives expected V T factor

1 dilatation mode $\phi_d = \partial_R \phi_b$

- Infinite Jacobian going to "collective coordinates"
- Integral over R gives infinity
- Infinities don't cancel – they multiply $Z = \infty^2$

How do we find the rest of the eigenvalues?

$$\left[-\Box - \frac{24R^2}{(R^2+r^2)^2}\right]\phi_j = \lambda_j\phi_j$$

Also need $-\Box\widehat{\phi}_j = \widehat{\lambda}_j\widehat{\phi}_j$

$$B = \frac{\det'(-\Box + \lambda\phi_b^2)}{\det(-\Box)} = \frac{\prod_j \lambda_j}{\prod_j \widehat{\lambda}_j}$$

det' = determinant with zero modes removed

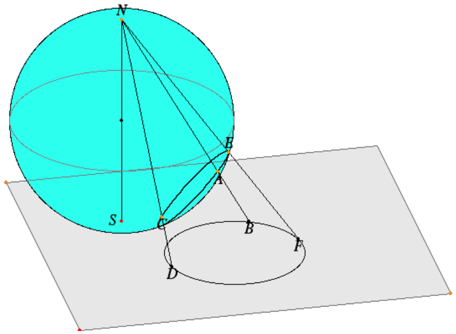
Stereographic projection

Andreassen, Frost, MDS (arXiv:1707.08124)

Rescale away constants

$$\mathcal{O} = -\Box - \frac{24R^2}{(R^2+r^2)^2} \xrightarrow{\text{R=1}} \mathcal{O} = -\Box - \frac{24}{(1+r^2)^2}$$

Stereographic mapping to 4-sphere



$$\left.\begin{aligned} \eta^\mu &= \frac{2x^\mu}{1+r^2} \\ \eta^5 &= \frac{1-r^2}{1+r^2} \end{aligned}\right\} \begin{aligned} \eta^2 &= 1 \\ \phi_b &= \eta^5 + 1 \end{aligned} \longrightarrow \mathcal{O}\cdot\phi_j = \phi_b(-\vec{L}_4^2 + 1)\cdot(\phi_b\phi_j)$$

- Angular momentum operator on 4-sphere
- Eigenfunctions are 4D spherical harmonics

$$\vec{L}_4^2 Y_{njlk} = \frac{1}{6}(n+1)(n+2)Y_{njlk}$$

$$B = \frac{\det'(-\Box + 3\lambda\phi_b^2)}{\det(-\Box)} = \frac{\det'(\vec{L}_4^2 - 1)}{\det(\vec{L}_4^2)} = \frac{\prod_n \left[-\frac{(n+1)(n+2)}{6} + 1\right]^{d_n}}{\prod_n \left[-\frac{(n+1)(n+2)}{6}\right]^{d_n}}$$

Degeneracy: $d_n = \frac{1}{6}(n+1)(n+2)(2n+3)$

Andreassen, Frost, MDS (arXiv:1707.08124)

Shortcut!

$$B = \frac{\det'(-\Box + 3\lambda\phi_b^2)}{\det(-\Box)} = \frac{\det(3\lambda\phi_b^2)}{\det(3\lambda\phi_b^2)} \times \frac{\det'\left(\frac{1}{3\lambda\phi_b^2}\Box - 1\right)}{\det\left(\frac{1}{3\lambda\phi_b^2}\Box\right)}$$

- Same eigenfunctions
- Eigenvalues shifted by 1

Note:
eigenfunctions of rescaled operator are **not** eigenfunctions of original operator:

$$\left(\frac{1}{3\lambda\phi_b^2}\Box - 1\right)\phi_j = \lambda_j\phi_j \Longrightarrow (-\Box + 3\lambda\phi_b^2)\,\phi_j = -\lambda_j 3\lambda\phi_b^2 \cdot \phi_j$$

Function not number

Explicitly, eigenfunctions are:

$$\phi_{nslm}(r,\alpha,\theta,\phi) = \frac{1}{r}P_{n+1}^{-s-1}\left(\frac{R^2-r^2}{R^2+r^2}\right)Y^{slm}(\alpha,\theta,\phi)$$

Degeneracy:

$$d_n = \frac{1}{6}(n+1)(n+2)(2n+3) = 1, 5, 14, 30, \cdots$$

$$\phi_{0000} = \sqrt{\frac{2}{\pi}}\frac{R}{R^2+r^2}$$

$$\phi_{1000} = \sqrt{\frac{2}{\pi}}R\frac{R^2-r^2}{(R^2+r^2)^2}$$

$$\phi_{2000} = \sqrt{\frac{2}{\pi}}\frac{R\,(r^4+R^4-3r^2R^2)}{(r^2+R^2)^3}$$

Eigenvalues of $\frac{1}{3\lambda\phi_b^2}\Box - 1$ are

$$\lambda_n^\phi = \lambda_{nslm} = \frac{(n-1)(n+4)}{6} = -\frac{2}{3}, 0, 1, \frac{7}{3}, \cdots$$

1 negative eigenvalue (tunneling)
5 zero eigenvalues (translation/dilitation)

Eigenvalues of $\frac{1}{3\lambda\phi_b^2}\Box$ are

$$\widehat{\lambda}_n^\phi = \lambda_n^\phi + 1 = \frac{(n+1)(n+2)}{6} = \frac{1}{3}, 1, 2, \frac{10}{3}, \cdots$$

Evaluate product

Andreassen, Frost, MDS (arXiv:1707.08124)

Product is UV divergent (large angular momentm n)

- Regulate with dimensional regularization
- Possible to evaluate exactly

$$B = \sqrt{\frac{\det'(-\Box + 3\lambda\phi_b^2)}{\det(-\Box)}} = \sqrt{\frac{\prod_n \left[-\frac{(n+1)(n+2)}{6} + 1\right]^{d_n}}{\prod_n \left[-\frac{(n+1)(n+2)}{6}\right]^{d_n}}} = \frac{25}{36}\sqrt{\frac{5}{6}}\exp\left[\frac{3}{2\varepsilon} - \frac{5}{4} + 6\zeta'(-1) + 3\ln\frac{R\mu}{2}\right]$$

Decay rate is then

$$\frac{\Gamma}{V} = \frac{1}{2TV}e^{-S[\phi_b]}\int d^4x \int dR J_T^4 J_d \sqrt{\frac{\det'(-\Box + 3\lambda\phi_b^2)}{\det(-\Box)}}$$

Jacobian for translation

$$J_T = \frac{1}{R}\sqrt{\frac{6S[\phi_b]}{5\pi}}$$

Jacobian for dilitation

$$J_d = \frac{1}{R}\sqrt{\frac{6S[\phi_b]}{5\pi}}$$

Finite after rescaling

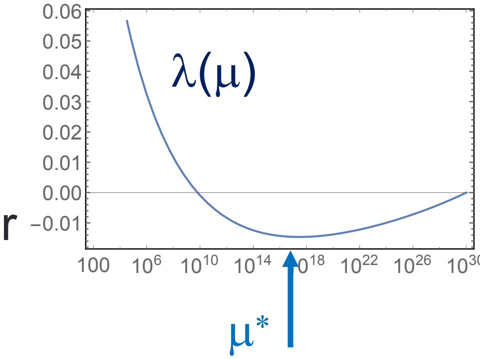
Still IR divergent (integral over R)

$$\frac{\Gamma}{V} = e^{-S[\phi_b]}\frac{1}{2}\left(\frac{6S[\phi_b]}{5\pi}\right)^{\frac{5}{2}}\int \frac{dR}{R^5}\frac{25}{36}\sqrt{\frac{5}{6}}\exp\left[\frac{3}{2\varepsilon} - \frac{5}{4} + 6\zeta'(-1) + 3\ln\frac{R\mu}{2}\right]$$

Integral over R

Andreassen, Frost, MDS (arXiv:1707.08124)

$$\frac{\Gamma}{V} = \int_0^\infty \frac{dR}{R^5} e^{\frac{1}{\hbar}\frac{8\pi^2}{3\lambda(\mu)} - \frac{8\pi^2}{3}\frac{\beta(\mu)}{\lambda(\mu)^2}\ln(\mu R)} (\cdots) = \infty$$



- Can't choose μ = R in integrand, since R is integrated over
- Can choose μ=μ* where β(μ*) = 0 (fixed point of λ)

$$\frac{\Gamma}{V} = e^{\frac{1}{\hbar}\frac{8\pi^2}{3\lambda(\mu)}} \int \frac{dR}{R^5} = \infty$$

- Knowing 1-loop exponent gives μ and R dependence at 2-loops

$$\Gamma_2 = \int_0^\infty \frac{dR}{R^5} e^{\hbar S[\phi_b^\star]\frac{\beta'_{0\star}}{\lambda_\star}\ln^2 R\mu^\star} = \mu_\star^4 \sqrt{-\frac{\pi\lambda_\star}{\hbar S[\phi_b^\star]\beta'_{0\star}}}\, e^{-\frac{4\lambda_\star}{\hbar S[\phi_b^\star]\beta'_{0\star}}}$$

Superleading dependence on $\hbar$

Step 4: Use perturbation theory properly

- All terms fixed by 1-loop RGEs important.
  - After some careful power-counting and resummation

$$\frac{\Gamma}{V} = \left[e^{-S[\phi_b]}\frac{1}{2}(RJ_T)^4(RJ_d)\mathrm{Im}\sqrt{\frac{\det \widehat{\mathcal{O}}_\phi}{\det' \mathcal{O}_\phi}} \right]_{R=\mu^{-1}=(\mu^\star)^{-1}} \times \mu_\star^4 \sqrt{-\frac{\pi\lambda_\star S[\phi_b^\star]}{\beta'_{0\star}}}\, e^{-\frac{4\lambda_\star}{\hbar S[\phi_b^\star]\beta'_{0\star}}} \left[\frac{\lambda_\star}{\lambda_{\text{1-loop}}(\hat{\mu})} - 1 - \frac{4\lambda_\star}{S[\phi_b^\star]^2\beta'_{0\star}}\right]$$

Gauge bosons

Quadratic fluctuations around the bounce in Fermi gauges: $\mathcal{L}_{\text{GF}} = \frac{1}{2\xi}(\partial_\mu A_\mu)^2$

$$\mathcal{L}_{\text{Fermi}} = \frac{1}{2}A_\mu\left[(-\Box + g^2\phi_b^2)\delta_{\mu\nu} + \frac{\xi-1}{\xi}\partial_\mu\partial_\nu\right]A_\nu + \frac{1}{2}G\left[-\Box + \lambda\phi_b^2\right]G \ + gA_\mu(\partial_\mu\phi_b)G - g\phi_b A_\mu\partial_\mu G - \bar{c}\Box c$$

- Bounce is spherically symmetric, use 3D spherical harmonics
- Radial fluctuations of longitudinal, scalar, and Goldstone modes couple
- Need

$$\frac{\det'\begin{pmatrix} -\Delta_s + \frac{3}{r^2} + g^2\phi_b^2 & -\frac{2\sqrt{s(s+2)}}{r^2} & g\phi_b' - g\phi_b\partial_r \\ -\frac{2\sqrt{s(s+2)}}{r^2} & -\Delta_s - \frac{1}{r^2} + g^2\phi_b^2 & -\frac{\sqrt{s(s+2)}}{r}g\phi_b \\ 2g\phi_b' + g\phi_b\partial_r + \frac{3}{r}g\phi_b & -\frac{\sqrt{s(s+2)}}{r}g\phi_b & -\Delta_s + \lambda\phi_b^2 \end{pmatrix}}{\det\begin{pmatrix} -\Delta_s + \frac{3}{r^2} & -\frac{2\sqrt{s(s+2)}}{r^2} & 0 \\ -\frac{2\sqrt{s(s+2)}}{r^2} & -\Delta_s - \frac{1}{r^2} & 0 \\ 0 & 0 & -\Delta_s \end{pmatrix}}$$

- Mapping to 4-sphere does not help
- Need new tricks to compute this ratio

Gelfand-Yanglom method

Want to compute $R = \frac{\det \mathcal{O}_1}{\det \mathcal{O}_2} = \frac{\prod_j \lambda_{1j}}{\prod_j \lambda_{2j}}$

1. Find zero-modes regular at r=0 $\mathcal{O}_1 \phi_1 = 0$ and $\mathcal{O}_2 \phi_2 = 0$
2. R is given by the simple formula $R = \left[\lim_{r\to 0} \frac{\phi_2(r)}{\phi_1(r)}\right] \times \left[\lim_{r\to\infty} \frac{\phi_1(r)}{\phi_2(r)}\right]$

That's it!

Example $R_s = \frac{\det\left[\frac{1}{3\lambda\phi_b^2}\Delta_s - 1\right]}{\det\left[\frac{1}{3\lambda\phi_b^2}\Delta_s\right]} = \prod_{n\geq s} \frac{\frac{1}{6}(n+1)(n+2) - 1}{\frac{1}{6}(n+1)(n+2)} = \frac{s(s-1)}{(s+2)(s+3)}$ ✓

$$\phi_1(r) = \frac{r^s}{(R^2+r^2)^2}\left(R^4 + \frac{2R^2(s-1)}{s+2} r^2 + \frac{s(s-1)}{(s+2)(s+3)} r^4\right)$$

$$\phi_2(r) = r^s$$

$\Longrightarrow$

$$\lim_{r\to 0} \frac{\phi_2(r)}{\phi_1(r)} = 1$$

$$\lim_{r\to\infty} \frac{\phi_1(r)}{\phi_2(r)} = \frac{s(s-1)}{(s+2)(s+3)}$$

$\Longrightarrow$

$$R_s = \frac{s(s-1)}{(s+2)(s+3)}$$ ✓

Integrate gauge boson fluctuations

$$\Delta_s \equiv \partial_r^2 + \frac{3}{r}\partial_r - \frac{s(s+2)}{r^2}$$

$$\begin{pmatrix} -\Delta_s + \frac{3}{r^2} + g^2\phi_b^2 & -\frac{2\sqrt{s(s+2)}}{r^2} & g\phi_b' - g\phi_b\partial_r \\ -\frac{2\sqrt{s(s+2)}}{r^2} & -\Delta_s - \frac{1}{r^2} + g^2\phi_b^2 & -\frac{\sqrt{s(s+2)}}{r} g\phi_b \\ 2g\phi_b' + g\phi_b\partial_r + \frac{3}{r}g\phi_b & -\frac{\sqrt{s(s+2)}}{r} g\phi_b & -\Delta_s + \lambda\phi_b^2 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = 0$$

- Need to find 3 linearly independent solutions
- Would have been impossible, but…
  - Don't need full solutions, just asymptotic behavior (large and small r)
  - Can simplify with auxiliary equations [Endo et al, 2017]

A lot of hard work

$$\kappa \equiv \sqrt{1 + \frac{8g^2}{\lambda}}$$

$$\frac{\det \mathcal{M}_s^{SLG}}{\det \widehat{\mathcal{M}}_s^{SLG}} = \frac{\det \widehat{\Psi}(0)}{\det \Psi(0)} \frac{\det \Psi(\infty)}{\det \widehat{\Psi}(\infty)} = C_\eta \frac{s}{s+2}$$

$$C_\eta = \frac{\Gamma(1+s)\Gamma(2+s)}{\Gamma(s+\frac{3}{2}-\frac{\kappa}{2})\Gamma(s+\frac{3}{2}+\frac{\kappa}{2})}$$

- Result is non-perturbative in g
- Still need to take product over spins s
- Still need to renormalize

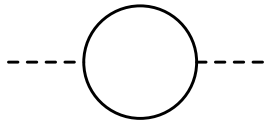
Renormalize

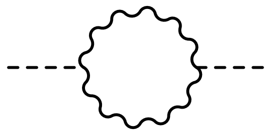
Subtract order g² terms

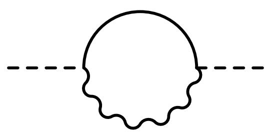
$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

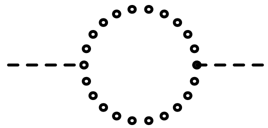
$$\left[\ln R^{SLG,R_\xi}_{s,\,\mathrm{diag}}\right]_{\mathrm{sub}} = \frac{2\left(s^3+4s^2+4s+2\right)}{s^2(s+1)^2} - 4\psi'(s)$$
$$+\left[\frac{2\left(s^4+9s^3+20s^2+18s+8\right)}{s^2(s+1)^2(s+2)} - 8\psi'(s)\right]\frac{g^2}{\lambda}$$
$$+\left[\frac{2(2s+5)\left(3s^4+12s^3+18s^2+12s+4\right)}{s^2(s+1)^2(s+2)^2} - 12\psi'(s)\right]\frac{g^4}{\lambda^2}$$

Add in g² terms with d-dimensional loops in MS

$$-S^{R_\xi}_{GG} =$$  $$= \left(\frac{g^2+\lambda}{\lambda}\right)^2\left[\frac{1}{6\varepsilon}+\frac{5}{18}+\frac{1}{6}\gamma_E+\frac{1}{6}\ln\pi R^2\mu^2\right]$$

$$-S^{R_\xi}_{AA} =$$  $$= \frac{g^4}{\lambda^2}\left[\frac{2}{3\varepsilon}+\frac{7}{9}+\frac{2}{3}\gamma_E+\frac{2}{3}\ln\pi R^2\mu^2\right]$$

$$-S^{R_\xi}_{AG} =$$  $$= -\frac{g^2}{\lambda}\left[\frac{4}{3\varepsilon}+\frac{26}{9}+\frac{4}{3}\gamma_E+\frac{4}{3}\ln\pi R^2\mu^2\right]$$

$$-S^{R_\xi}_{\bar{c}c} =$$  $$= \frac{g^4}{\lambda^2}\left[-\frac{1}{3\varepsilon}-\frac{5}{9}-\frac{1}{3}\gamma_E-\frac{1}{3}\ln\pi R^2\mu^2\right]$$

Final answer

THE LIFETIME OF THE STANDARD MODEL

Decay rate in Standard Model

Final formula

$$\frac{\Gamma}{V} = e^{-S[\phi_b]}\frac{1}{2}\mathrm{Im}\,V_{SU(2)}J_G^3(RJ_T)^4(RJ_d)\sqrt{\frac{\det\widehat{\mathcal{O}}_h}{\det'\mathcal{O}_h}}\sqrt{\frac{\det\widehat{\mathcal{O}}_{ZG}}{\det'\mathcal{O}_{ZG}}}\frac{\det\widehat{\mathcal{O}}_{WG}}{\det'\mathcal{O}_{WG}}\sqrt{\frac{\det\mathcal{O}_{\bar{t}t}}{\det\widehat{\mathcal{O}}_{\bar{t}t}}}\sqrt{\frac{\det\mathcal{O}_{\bar{b}b}}{\det\widehat{\mathcal{O}}_{\bar{b}b}}}$$
$$\times\,\mu_\star^4\sqrt{-\frac{\pi S[\phi_b^\star]\lambda_\star}{\beta'_{0\star}}}\,e^{-\frac{4\lambda_\star}{S[\phi_b^\star]\beta'_{0\star}}}\left[\frac{\lambda_\star}{\lambda_{1\text{-loop}}(\hat{\mu})}-1-\frac{4\lambda_\star}{S[\phi_b^\star]^2\beta'_{0\star}}\right] \quad (6.2)$$

Everything known analytically:

$$\sqrt{\frac{\det\widehat{\mathcal{O}}_{ZG}}{\det'\mathcal{O}_{ZG}}} = \exp\left[-\frac{3}{2}S_{\text{fin}}\left(-\frac{g_Z^2}{12\lambda}\right)-\frac{1}{2}S_{\text{diff}}^{AG}\left(-\frac{g_Z^2}{12\lambda}\right)-S_{\text{loops}}^{AG}\left(-\frac{g_Z^2}{12\lambda}\right)-\frac{1}{2}S_{\text{fin}}^{G}\right]$$

$$\frac{\det\widehat{\mathcal{O}}_{WG}}{\det'\mathcal{O}_{WG}} = \exp\left[-3S_{\text{fin}}\left(-\frac{g_W^2}{12\lambda}\right)-S_{\text{diff}}^{AG}\left(-\frac{g_W^2}{12\lambda}\right)-2S_{\text{loops}}^{AG}\left(-\frac{g_W^2}{12\lambda}\right)-S_{\text{fin}}^{G}\right]$$

$$S_{\text{fin}}(x) = (-3+6\gamma_E)x^2+\frac{11}{36}+\ln 2\pi+\frac{3}{4\pi^2}\zeta(3)-4\zeta'(-1)$$
$$-x\kappa_x\left[\psi^{(-1)}\left(\frac{3+\kappa_x}{2}\right)-\psi^{(-1)}\left(\frac{3-\kappa_x}{2}\right)\right]+\left(6\,x-\frac{1}{6}\right)\left[\psi^{(-2)}\left(\frac{3+\kappa_x}{2}\right)+\psi^{(-2)}\left(\frac{3-\kappa_x}{2}\right)\right]$$
$$+\,\kappa_x\left[\psi^{(-3)}\left(\frac{3+\kappa_x}{2}\right)-\psi^{(-3)}\left(\frac{3-\kappa_x}{2}\right)\right]-2\left[\psi^{(-4)}\left(\frac{3+\kappa_x}{2}\right)+\psi^{(-4)}\left(\frac{3-\kappa_x}{2}\right)\right] \quad (4.8)$$

...

$$\sqrt{\frac{\det\mathcal{M}_{\bar{t}t}}{\det\widehat{\mathcal{M}}_{\bar{t}t}}} = \exp\left[\frac{N_C}{2}S_{\text{fin}}^{\bar\psi\psi}\left(\sqrt{\frac{y_t^2}{\lambda}}\right)-N_C S_{\text{loops}}^{\bar\psi\psi}\left(\frac{y_t^2}{\lambda}\right)\right]$$

Numerical results

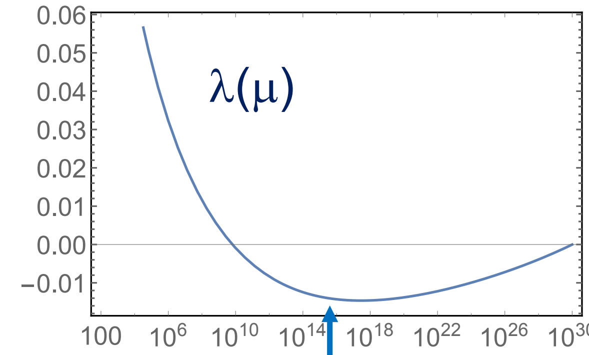
Inputs with negligible uncertainty:

$G_F = 1.115\times10^{-5}\mathrm{GeV}^{-2}; \quad m_W^{\rm pole} = 80.385\ \mathrm{GeV}, \quad m_Z^{\rm pole} = 91.1876\ \mathrm{GeV}, \quad m_b^{\rm pole} = 4.93\ \mathrm{GeV}$

Inputs with non-negligible uncertainty:

$m_t^{\rm pole} = 173.1 \pm 0.6\ \mathrm{GeV} \quad m_h^{\rm pole} = 125.09 \pm 0.24\ \mathrm{GeV} \quad \alpha_s(m_Z) = 0.1181 \pm 0.0011$

- Convert to MS at weak scale
  - 2-loop threshold corrections
  - 2-loop electroweak/strong mixed contributions
  - 3 and 4 loop strong contributions
  - 3-loop running for all, with 4-loops for α_s



Scale where $\beta_\lambda(\mu^*) = 0$ is $\mu_\star = 3.11 \times 10^{17}\ \mathrm{GeV}$

Quartic at μ^*:

$\lambda(\mu_\star) = -0.0138$

$\Gamma \sim e^{-\frac{8\pi^2}{\lambda(\mu^\star)}} \sim e^{-800}$

- Sets units for decay

$\Gamma \sim (\mu^\star)^4 \sim 10^{70}\ \mathrm{GeV}^4$

All order ~ 100 in exponent

Compare to lifetime of the universe

$H_0 = 67.4\frac{\mathrm{km}}{\mathrm{s\ Mpc}} = 1.44 \times 10^{-42}\ \mathrm{GeV}$

$H_0^4 = 10^{-168}\ \mathrm{GeV}^4$

Decay rate of our universe

Andreassen, Frost, MDS (arXiv:1707.08124)

Put all the factors in

$$\frac{\Gamma}{V} = \underbrace{e^{-S[\phi_b]}}_{10^{-826}} \underbrace{V_{SU(2)}}_{10^2} \underbrace{J_G^3}_{10^5} \underbrace{(RJ_T)^4 (RJ_d)}_{10^7} \underbrace{\sqrt{\frac{\det \widehat{\mathcal{O}}_h}{\det' \mathcal{O}_h}}}_{10^{-2}} \underbrace{\sqrt{\frac{\det \widehat{\mathcal{O}}_{ZG}}{\det' \mathcal{O}_{ZG}}}}_{10^{17}} \underbrace{\frac{\det \widehat{\mathcal{O}}_{WG}}{\det' \mathcal{O}_{WG}}}_{10^{19}} \underbrace{\sqrt{\frac{\det \mathcal{O}_{\bar{t}t}}{\det \widehat{\mathcal{O}}_{\bar{t}t}}}}_{10^{25}} \underbrace{\sqrt{\frac{\det \mathcal{O}_{\bar{b}b}}{\det \widehat{\mathcal{O}}_{\bar{b}b}}}}_{0.995}$$

$$\times \underbrace{\mu_\star^4}_{10^{70}\ \mathrm{GeV}^4} \underbrace{\sqrt{-\frac{\pi\lambda_\star}{S[\phi_b^\star]\beta'_{0\star}} e^{-\frac{4\lambda_\star}{S[\phi_b^\star]\beta'_{0\star}}}}}_{1.09} \underbrace{S[\phi_b^\star]\left[\frac{\lambda_\star}{\lambda_{\text{1-loop}}(\hat{\mu})} - 1 - \frac{4\lambda_\star}{S[\phi_b^\star]^2\beta'_{0\star}}\right]}_{0.653}$$

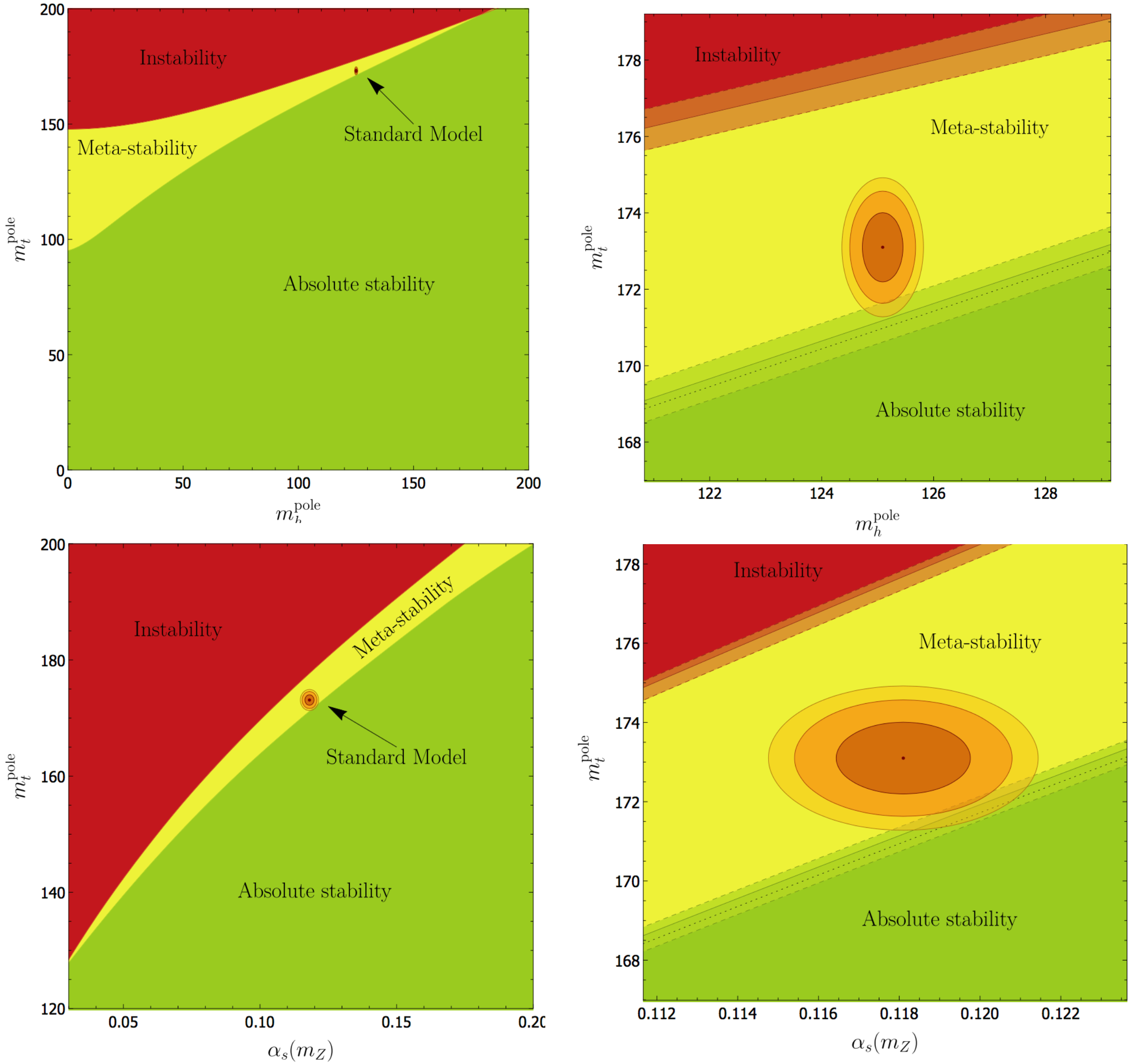
$$= 10^{-683}\,\mathrm{GeV}^4 \times \begin{pmatrix} 10^{-279} \\ 10^{162} \end{pmatrix}_{m_t} \times \begin{pmatrix} 10^{-39} \\ 10^{35} \end{pmatrix}_{m_h} \times \begin{pmatrix} 10^{-186} \\ 10^{127} \end{pmatrix}_{\alpha_s} \times \begin{pmatrix} 10^{-61} \\ 10^{102} \end{pmatrix}_{\text{thr.}} \times \begin{pmatrix} 10^{-2} \\ 10^{2} \end{pmatrix}_{\text{NNLO}}$$

$$= 10^{-683^{-409}_{+202}}\,\mathrm{GeV}^4$$

$$\tau_{\mathrm{SM}} = 10^{139^{+102}_{-51}}\ \text{years}$$

Stability phase diagram

Andreassen, Frost, MDS (arXiv:1707.08124)



$$\frac{\Gamma}{V} = 10^{-683^{-409}_{+202}}\ \mathrm{GeV}^4$$

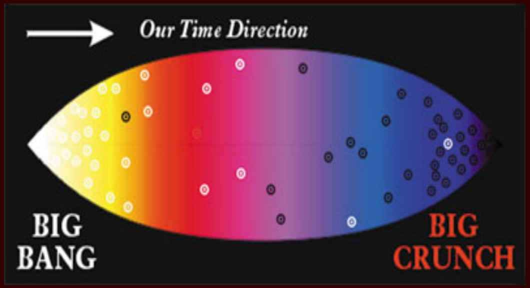
$$\tau_{\rm SM} = 10^{139^{+102}_{-51}}\ \text{years}$$

Top mass and α_s biggest experimental uncertainties

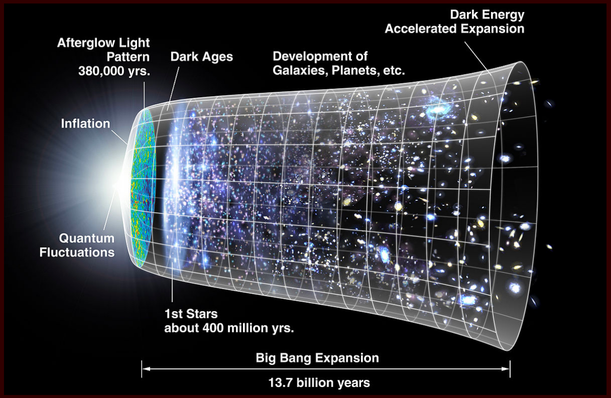
Conclusions: the fate of the universe

1. Static universe

2. Big Crunch

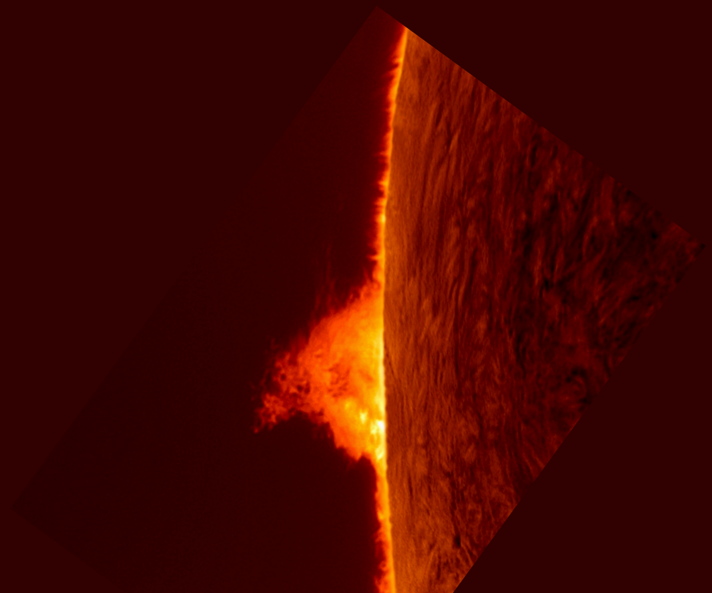


3. Cold and Empty Future



4. Vacuum decay

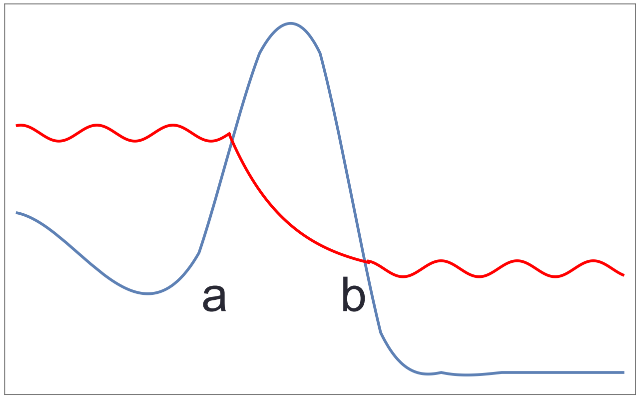
- Universe is infinite
- Decay rate is finite
- Somewhere a bubble of true vacuum has formed (10^{139} light years away)
- Wall is barreling towards us at the speed of light



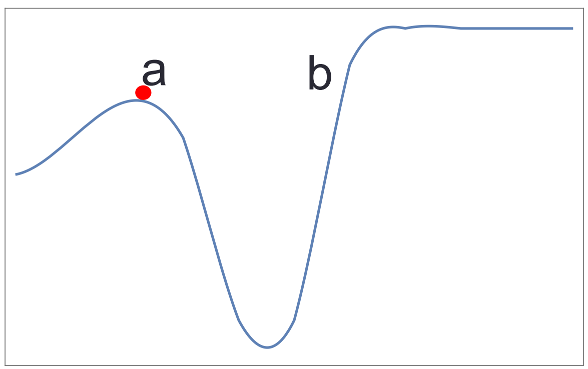
DERIVING THE DECAY RATE FORMULA

Colelman & Callan (1977)

Tunneling and WKB



Invert potential



Coleman (Erice p.266): "Every child knows that the amplitude for transmission obeys the WKB formula"

$$T = \exp\left\{ - \int_a^b dx \sqrt{2(V-E)} \right\}$$

Coleman 1977

- T can be written as $e^{-S_E[x_b]}$

"bounce" path $x_b(\tau)$

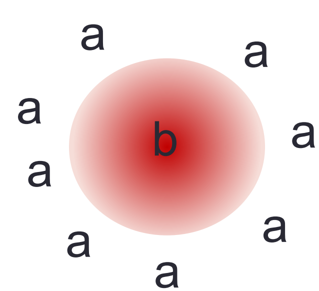
- Solution to equations of motion with "inverted" potential –V(x)
- Starts at a, ends at a, passes through b

Rate $\Gamma \propto T = e^{-S[x_b]}$

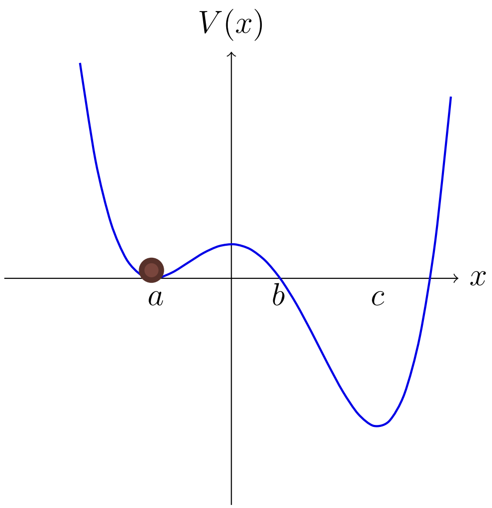
QFT: Rate $\Gamma \propto e^{-S_E[\phi_b]}$

QFT bounce has $\phi_b(\vec{x}) = a$ at $|\vec{x}| = \infty$

$\phi_b(0) = b$



Tunneling from path integral



Coleman and Callan 1977

$$Z \equiv \langle a | e^{-H\mathcal{T}} | a \rangle = \int_{x(0)=a}^{x(\mathcal{T})=a} \mathcal{D}x e^{-S_E[x]}$$

$$Z = \sum_E e^{-ET} |\psi_E(a)|^2$$

Isolate ground state energy from late times

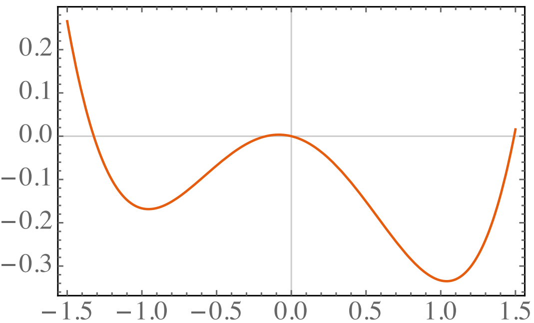
$$E_0 = -\lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \ln Z$$

Decay rate is the imaginary part of the energy

$$\frac{\Gamma}{2} = \text{Im} \lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \ln Z$$

Saddle points

$$Z \equiv \langle a|e^{-H\mathcal{T}}|a\rangle = \int_{x(0)=a}^{x(\mathcal{T})=a} \mathcal{D}x e^{-S_E[x]}$$



Dominated by **saddle points**
= solutions to the Euclidean equations of motion

$$S = \int dt \left[\frac{1}{2}(\partial_t x) - V(x)\right]$$

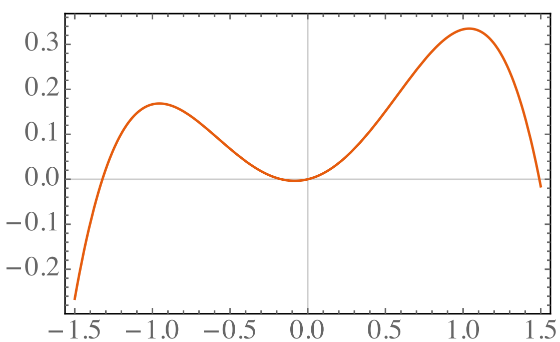
$$\partial_t^2 x = -V'(x)$$

$$S_E = \int d\tau \left[\frac{1}{2}(\partial_\tau x) + V(x)\right]$$

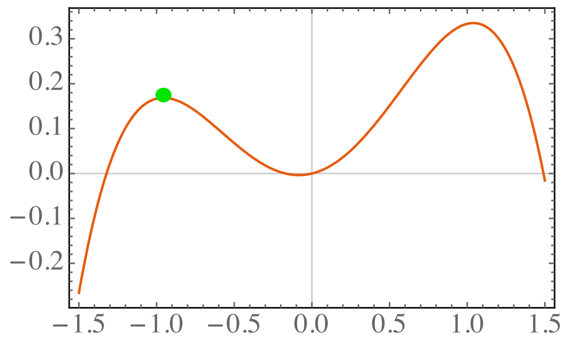
$$\partial_\tau^2 x = V'(x)$$

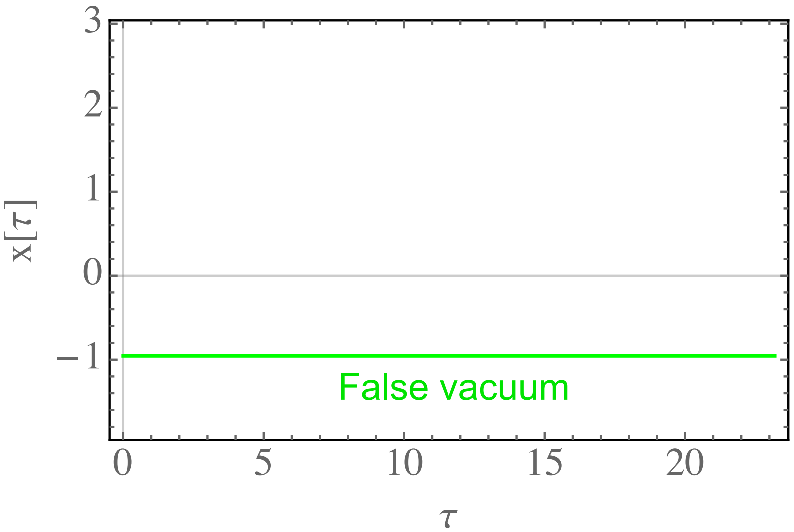
particle rolling down the inverted potential

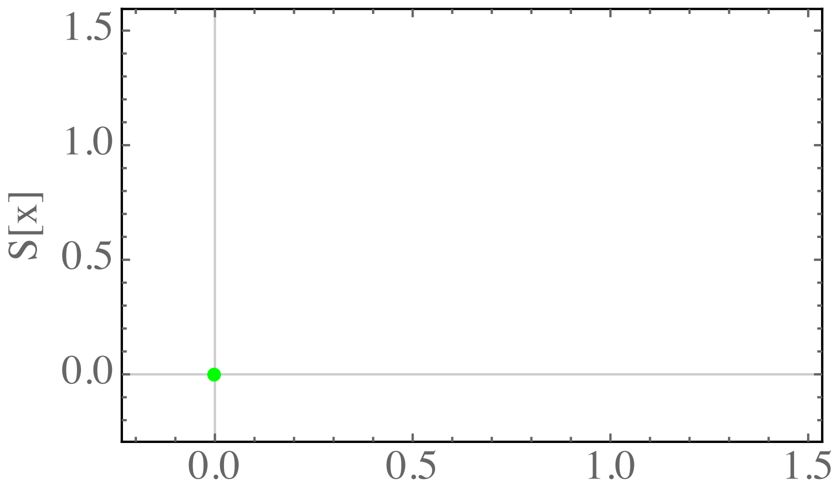
with boundary conditions $x(0) = x(\mathcal{T}) = a$



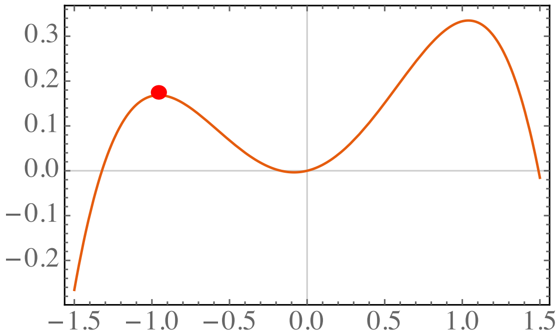
Saddle points

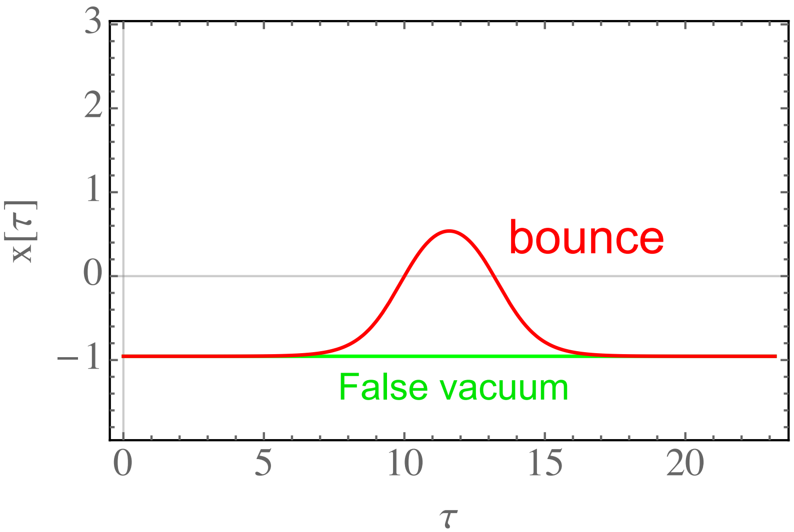


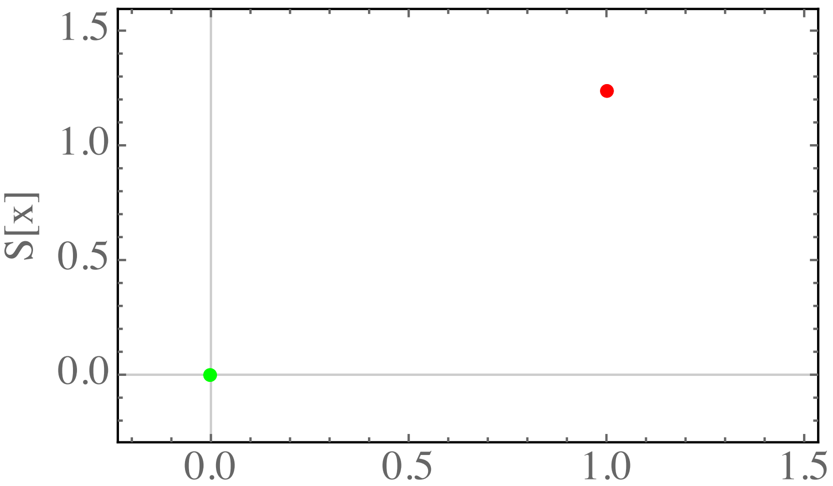




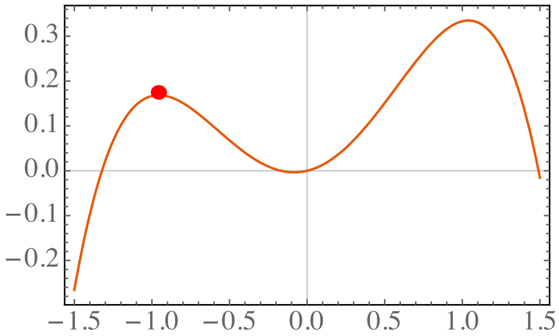
Saddle points

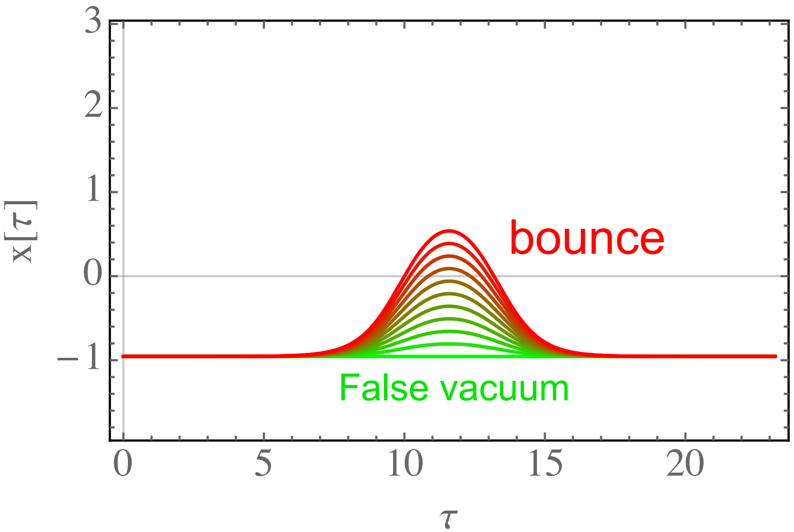




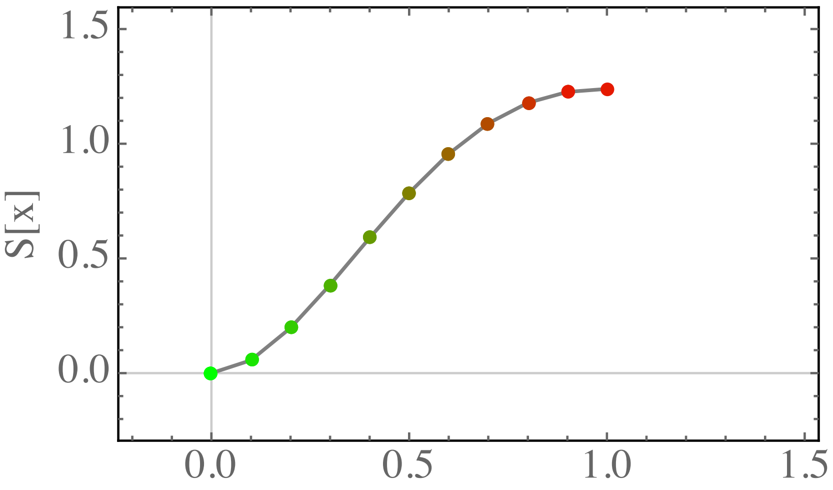


Saddle points





Bounce is a saddle point of action:
local maximum along one direction



Maximum → negative eigenvalue of S'' → Z has an imaginary part

$$Z = \int_{x(0)=a}^{x(\mathcal{T})=a} \mathcal{D}x e^{-S_E[x]}$$

$$= \int_{x(0)=0}^{x(\mathcal{T})=0} \mathcal{D}x e^{-S_E[\bar{x}]-\frac{1}{2}S_E''[\bar{x}]x^2-\cdots}$$

$$= e^{-S_E[\bar{x}]} \int_{x(0)=0}^{x(\mathcal{T})=0} \mathcal{D}x e^{-\frac{1}{2}\int d\tau\left\{-x\partial_\tau^2 x + xV''(\bar{x})\,x\right\}}$$

$$= \int d\xi_0 \cdots d\xi_n e^{\left[-\sum_n \frac{1}{2}\lambda_n \xi_n^2\right]}$$

$$= \sqrt{\frac{2\pi}{\lambda_1}}\sqrt{\frac{2\pi}{\lambda_2}}\cdots$$

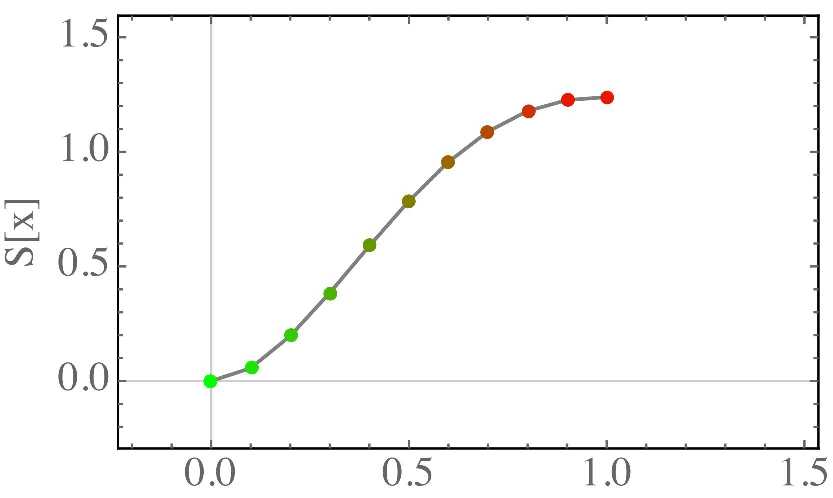
$$x(\tau) = \sum \xi_n y_n(\tau)$$

$$\left[-\partial_\tau^2 + V''(\bar{x})\right] y_n = \lambda_n y_n$$

$$\int d\tau y_n y_m = \delta_{nm}$$

One of these is **negative ($\lambda_1 < 0$)** if $\bar{x}$ is a maximum of S in some direction

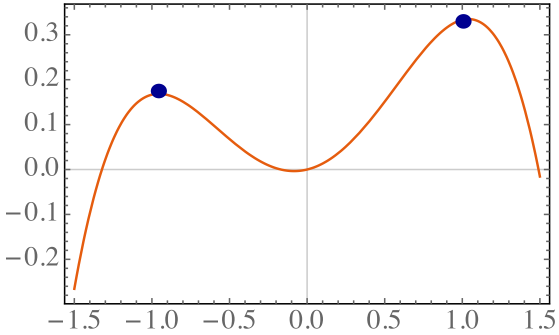
$$\frac{\Gamma}{2} = \text{Im} \lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \ln Z \neq 0$$

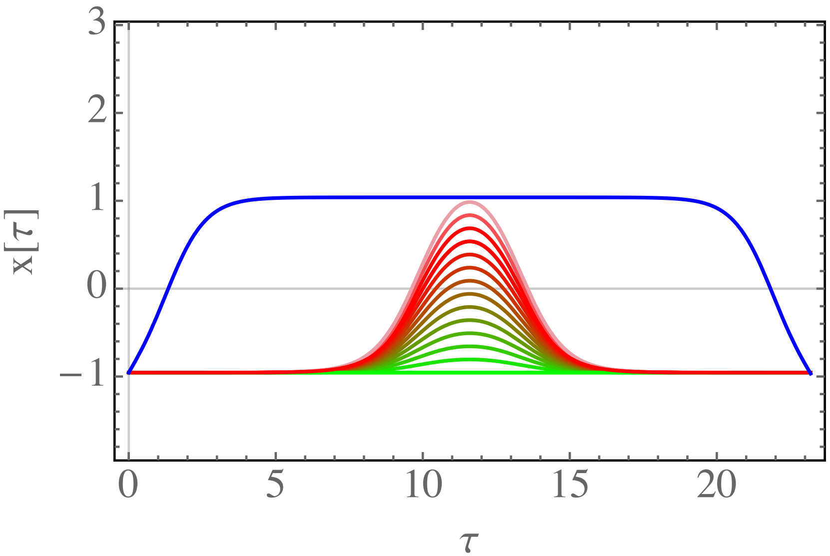


But Z is real! So how did this happen?

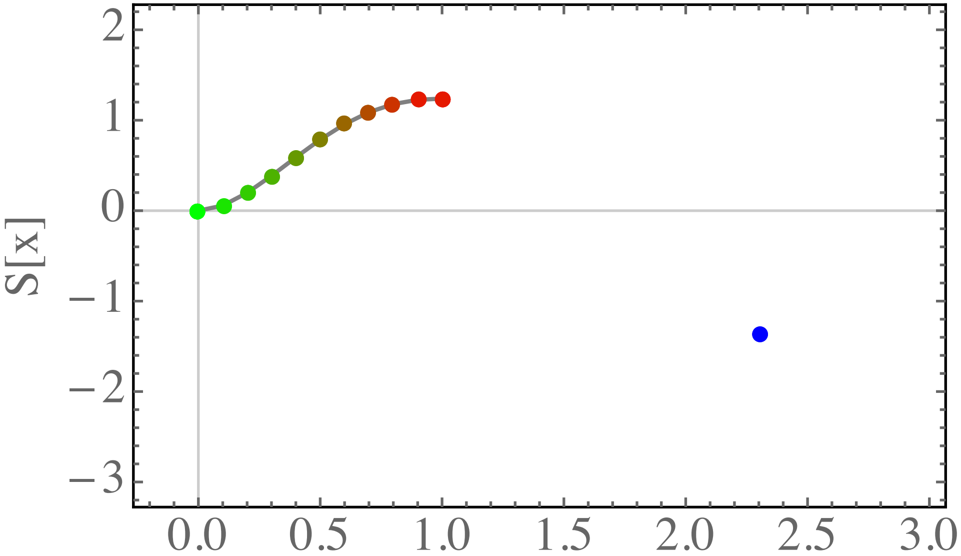
$$Z \equiv \langle a|e^{-H\mathcal{T}}|a\rangle = \int_{x(0)=a}^{x(\mathcal{T})=a} \mathcal{D}x e^{-S_E[x]}$$

The Shot

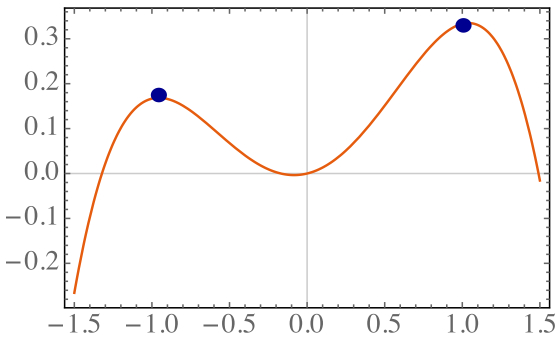


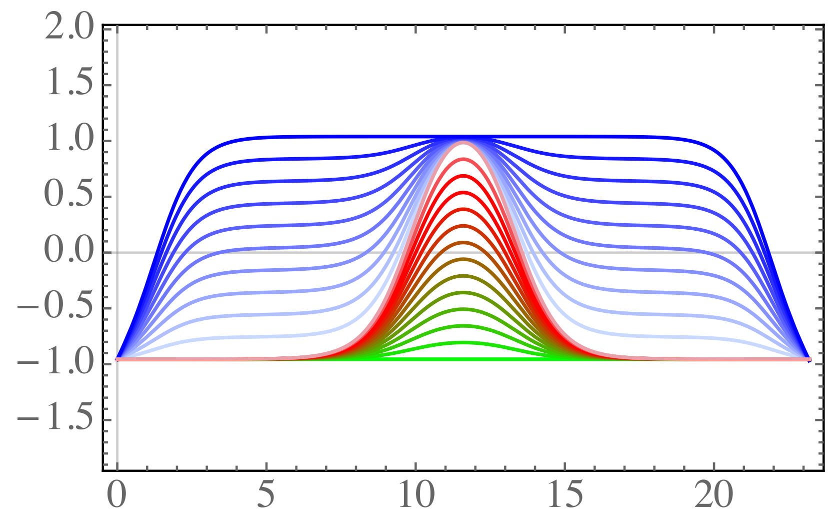


Shot stays at true vacuum most of the time



The Shot



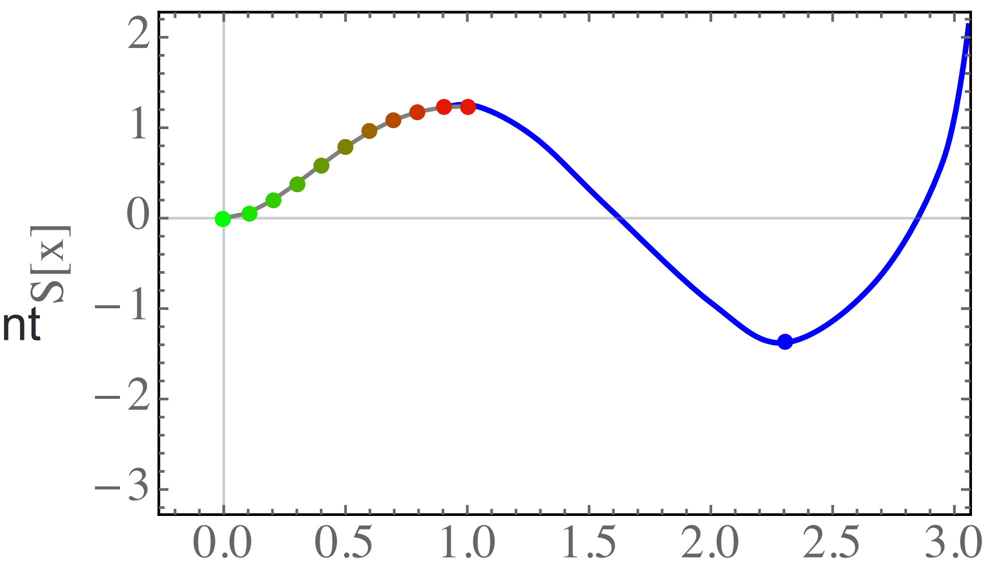


Shot stays at true vacuum most of the time

$$Z \equiv \langle a|e^{-H\mathcal{T}}|a\rangle = \int_{x(0)=a}^{x(\mathcal{T})=a} \mathcal{D}x e^{-S_E[x]}$$

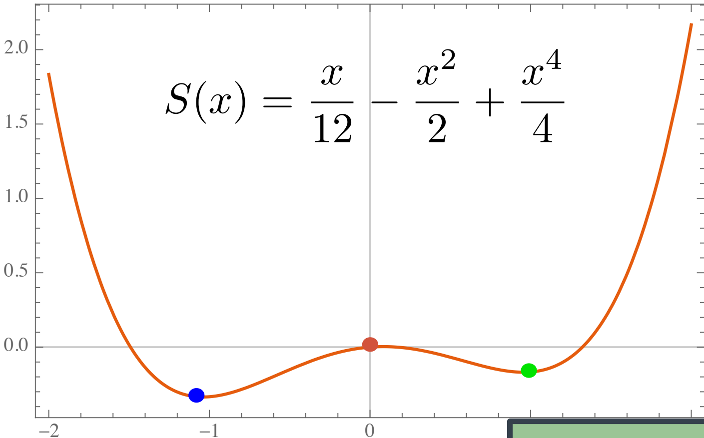
$$\approx e^{-S_E[x_{\text{shot}}]} \left(\gg e^{-S_E[x_{\text{bounce}}]}\right)$$

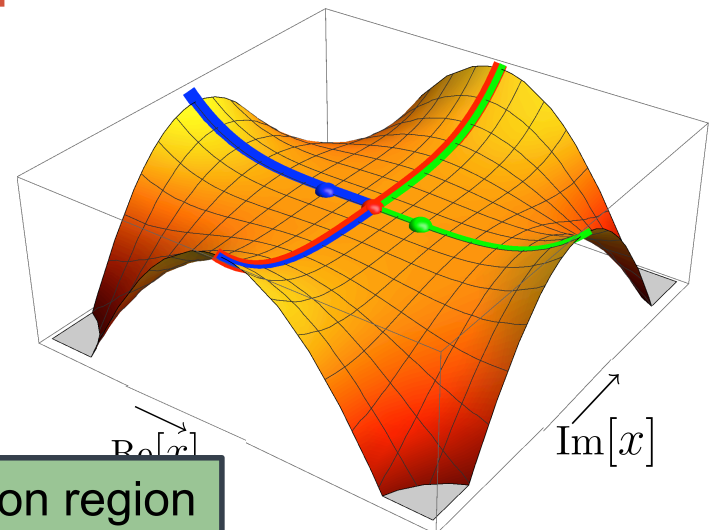
$$= e^{-E_0\mathcal{T}}$$



- **Bounce** is exponentially subdominant
- Consistent expansion must drop it
- **True vacuum dominates**

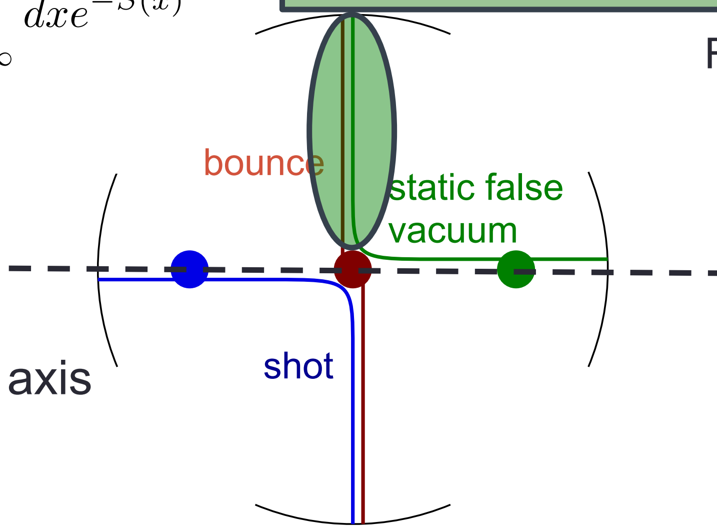
Contour integration





Need this integration region to get the "right" answer

$$Z = \int_{-\infty}^{\infty} dx e^{-S(x)}$$



Integrate along real axis

Real axis = sum of steepest descent contours

$$\int_{\mathbb{R}} = \int_{C_{\text{shot}}} + \int_{C_{\text{bounce}}} + \int_{C_{\text{FV}}}$$

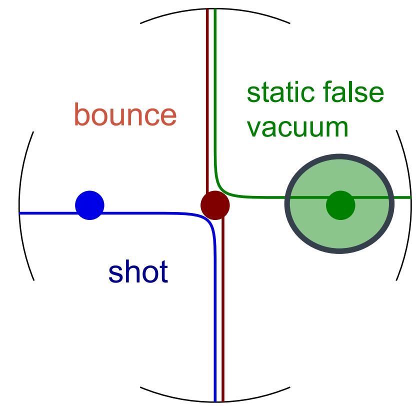
$$= e^{-S(x_{\text{shot}})} + i\frac{\Gamma}{2}\mathcal{T}$$

$$+ \; e^{-S(x_{\text{bounce}})} - i\Gamma\mathcal{T}$$

$$+ \; e^{-S(x_{\text{FV}})} + i\frac{\Gamma}{2}\mathcal{T}$$

$$\approx e^{-S(x_{\text{shot}})}$$

Discontinuity



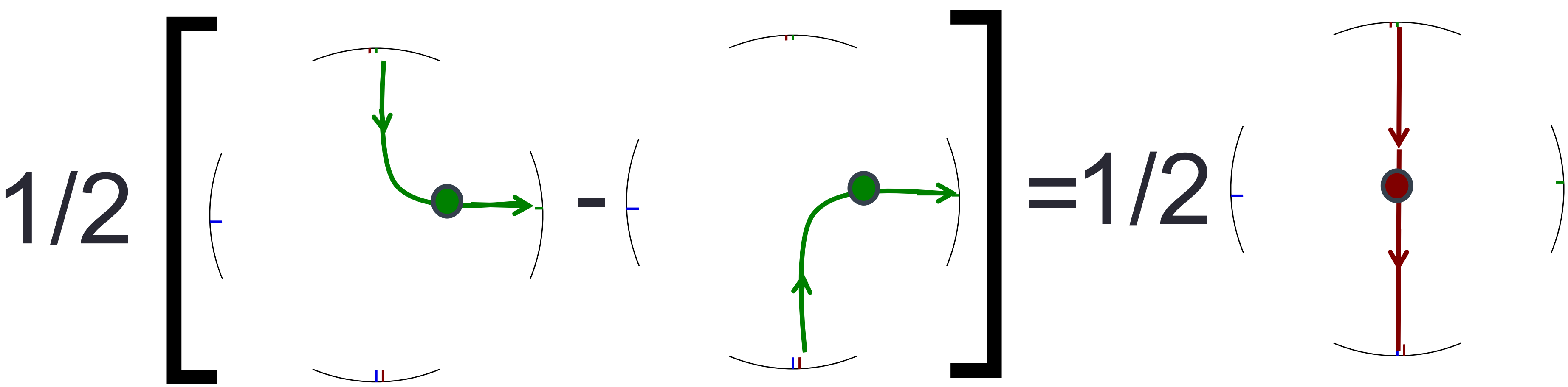
Can we just integrate along the FV contour?

Yes, at least for this toy integral

$$Z = \int_C dx e^{-S(x)}$$

No

- Not clear what "fixing to a contour" means for a path integral
- Saddle point approximation loses the imaginary part
  - Expanding around the saddle gives a real integral
  - Imaginary part comes from region far away
- Saddle point approximation does work for the discontinuity

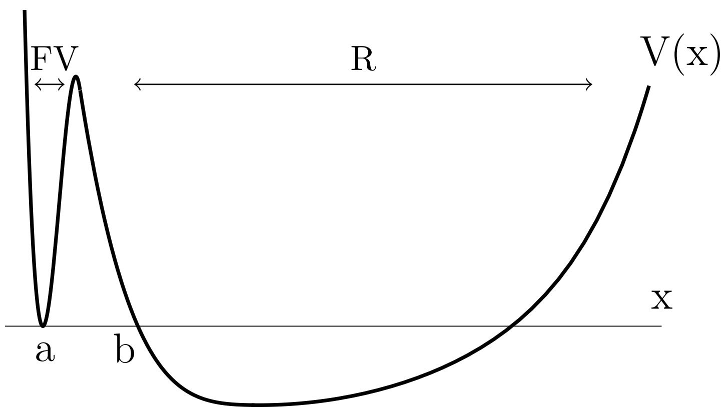


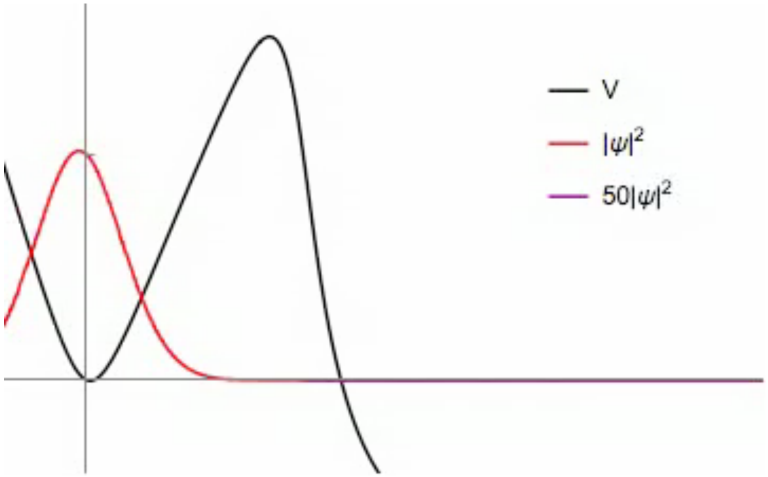
3. DIRECT METHOD

Andreassen, Farhi, Frost, MDS (2016)

Quantum mechanics

$$i\partial_t \psi(x,t) = \left[-\frac{1}{2m}\partial_x^2 + V(x)\right]\psi(x,t)$$

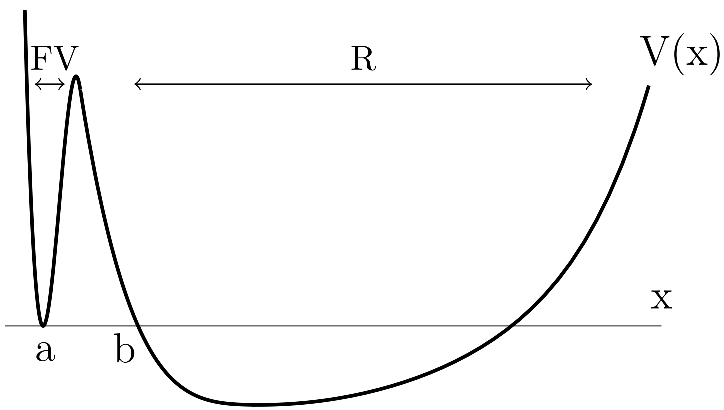


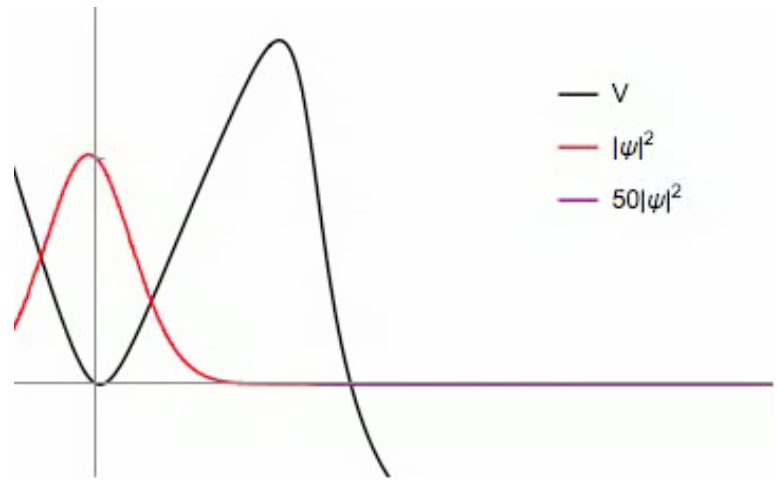


$$P_{\mathrm{FV}}(T) \equiv \int_{\mathrm{FV}} dx\, |\psi(x,T)|^2$$

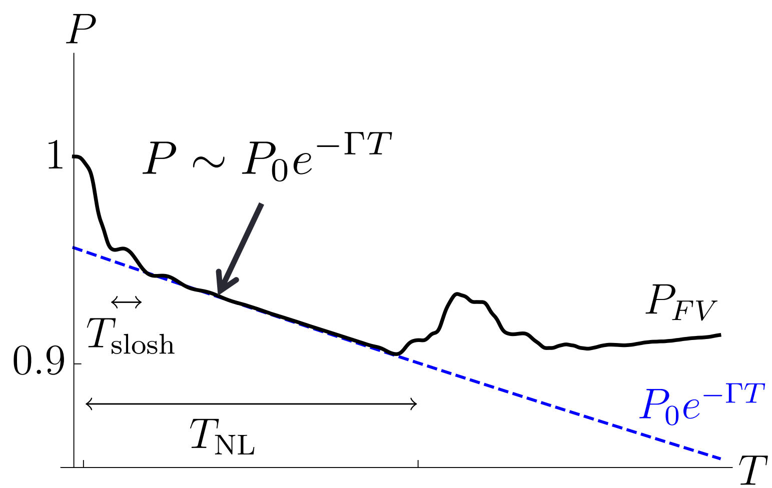
Quantum mechanics

$$i\partial_t \psi(x,t) = \left[-\frac{1}{2m}\partial_x^2 + V(x)\right]\psi(x,t)$$





$$P_{\rm FV}(T) \equiv \int_{\rm FV} dx\, |\psi(x,T)|^2$$

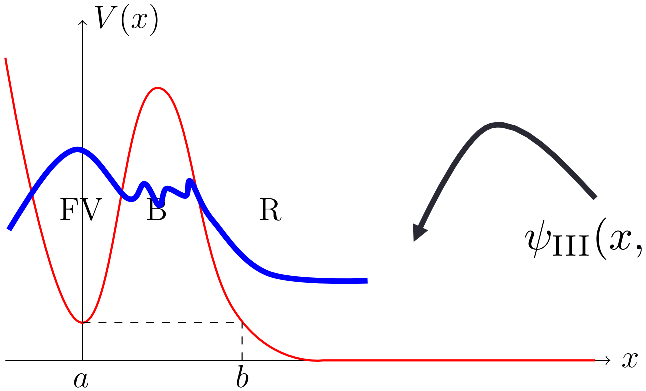


Two time scales

- $T > T_{\rm slosh}$ – removes transients
- $T < T_{\rm NL}$ -- avoids all ψ in true vacuum

$$\Gamma = -\lim_{\frac{T}{T_{\rm slosh}}\to\infty}\ \lim_{\frac{T}{T_{\rm NL}}\to 0} \frac{1}{P_{\rm FV}}\frac{d}{dT}P_{\rm FV}$$

Gamow's method



- Hermitian Hamiltonian → energies are real
  → $\psi^*\psi$ independent of time

$$\psi_{\rm III}(x,t) = C e^{i(kx-Et)} + D e^{-i(kx-Et)}$$

Enforces $T \ll T_{NL}$ (no return flux)

Choose outgoing boundary conditions: D=0, $\psi_{\rm III}(x,t) = C e^{i(kx-Et)}$

- Modes now have outgoing flux

$$J = i(\psi^\star \partial_x \psi - \psi \partial_x \psi^\star) = -2p$$

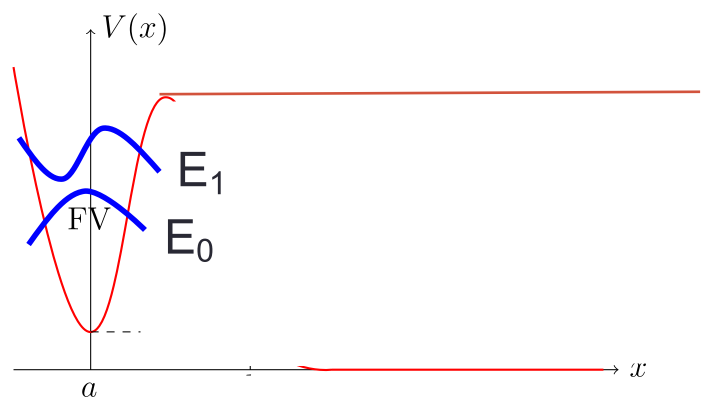
- Violates unitarity → energies are complex

$$E = E_0 - \frac{i}{2}\Gamma \qquad \psi(x,t) = e^{-iE_0 t - \frac{1}{2}\Gamma t}\psi_0(x)$$

- Probability is time dependent

$$P = \int \psi^\star \psi \sim e^{-\Gamma t}$$

- Zeros of D → energies are quantized
  - Resonances ~ bound states



Assume E_1 E_2 etc components already died off

Enforces $T \gg T_{slosh}$ (only metastable FV decay)

A direct approach

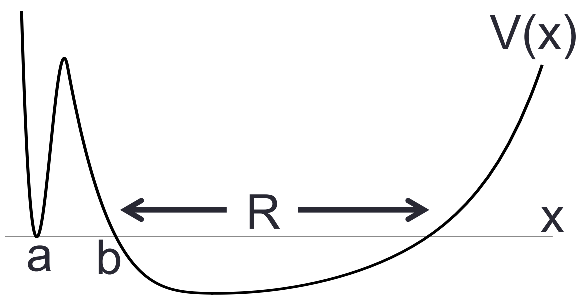
$$\Gamma = -\lim_{\frac{T}{T_{\text{slosh}}}\to\infty}\ \lim_{\frac{T}{T_{\text{NL}}}\to 0} \frac{1}{P_{\text{FV}}}\frac{d}{dT}P_{\text{FV}}$$

T >> T_{slosh} (only metastable FV decay)

T << T_{NL} (no return flux)

Back to our definition

$$P_{\text{FV}}(T) \equiv \int_{\text{FV}} dx\, |\psi(x,T)|^2$$



- Start with: $\psi(x, t=0) = \delta(x-a)$
- We will compute

$$\Gamma_{\text{R}} \equiv \lim_{\substack{T/T_{\text{NL}}\to 0 \\ T/T_{\text{slosh}}\to\infty}} \frac{1}{P_{\text{FV}}}\frac{dP_{\text{R}}}{dT}$$

Propagator from a to x_f in time T

$$D(a, x_f, T) \equiv \int_{x(0)=a}^{x(T)=x_f} \mathcal{D}x\, e^{iS[x]}$$

$$P_R(T) = \int_R dx_f\, |D(a, x_f, T)|^2$$

probability of finding ψ
in region R at time T

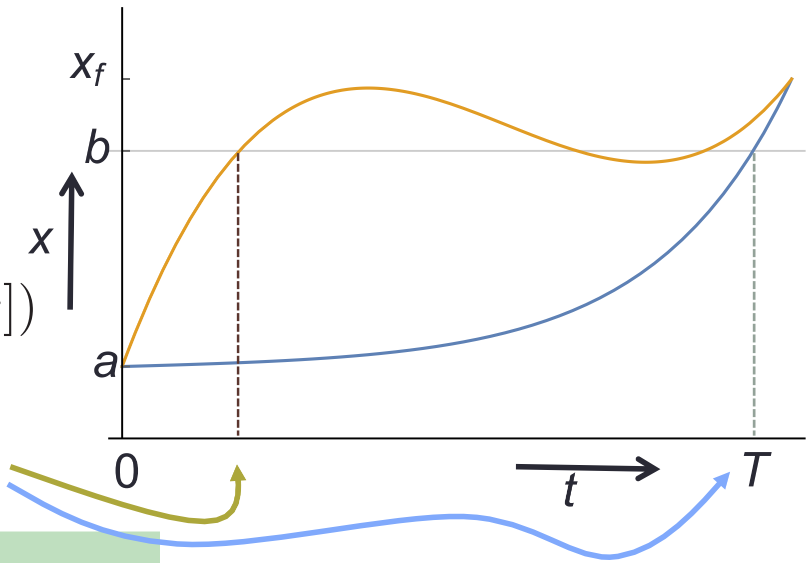
Step 1: Split up propagator

$$D(a, x_f, T) \equiv \int_{x(0)=a}^{x(T)=x_f} \mathcal{D}x\, e^{iS[x]}$$

Split path integral into *before b* and *after b*:

$$D(a, x_f, T) = \int_{x(0)=a}^{x(T)=x_f} \mathcal{D}x\, e^{iS[x]} \int dt\, \delta(t - t_b[x])$$

$t_b[x] \equiv$ First time path *x(t)* hits *b*



$$= \int dt \underbrace{\int_{x(0)=a}^{x(t)=b} \mathcal{D}x\, e^{iS[x]} \delta(t - t_b[x])}_{\bar{D}(a,b,t)} \underbrace{\int_{x(t)=b}^{x(T)=x_f} \mathcal{D}x\, e^{iS[x]}}_{D(b, x_f, T-t)}$$

hits b only once, at t

- Regular propagator from b to x_f
- Paths can go back past b

$$D(a, x_f, T) = \int dt\, \bar{D}(a, b, t) D(b, x_f, T - t)$$

Step 2: Apply T << T_{NL}

$$D(a,x_f,T)=\int dt\,\bar{D}(a,b,t)D(b,x_f,T-t)$$

- Hits b only once, at t

- Regular propagator from b to x_f
- Paths can go back past b

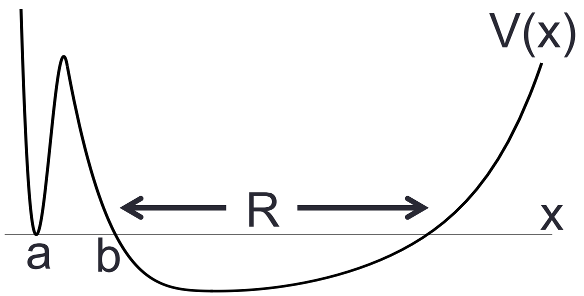
$$P_R(T)=\int_R dx_f\,|D(a,x_f,T)|^2$$
$$=\int dx_f dt_1 dt_2 \bar{D}(a,b,t_1)\bar{D}^*(a,b,t_2)\underbrace{D(b,x_f,T-t_1)}\underbrace{D(b,x_f,T-t_2)}$$
$$\langle x_f,T|b,t_1\rangle\langle b,t_2|x_f,T\rangle$$

T << T_{NL} (no return flux)

Propagation from *b* out of R is negligible: $\int_R dx_f\,|x_f\rangle\langle x_f|=1$

$$P_R(T)=\int dt_1 dt_2 \bar{D}(a,b,t_1)\bar{D}^*(a,b,t_2)\langle b,t_2|b,t_1\rangle$$
$$=\int_0^T dt D(a,b,t)\bar{D}^*(a,b,t)+\text{c.c.}$$

Step 3: Simplify



$$P_R(T) = \int_0^T dt D(a,b,t)\bar{D}^*(a,b,t) + \text{c.c.}$$

$$\Gamma_\text{R} \equiv \lim_{\substack{T/T_\text{NL}\to 0 \\ T/T_\text{slosh}\to\infty}} \frac{1}{P_\text{FV}}\frac{dP_\text{R}}{dT} = \lim_{T\to\infty} \frac{D(a,b,T)\bar{D}^*(a,b,T)}{\int_\text{FV} dx\, |D(a,x,T)|^2} + \text{c.c.}$$

Go to Euclidean time and take $T >> T_{slosh}$

$$\Gamma_R = 2\text{Im}\lim_{T\to\infty}\left(\frac{\int_{x(-\mathcal{T})=a}^{x(\mathcal{T})=a}\mathcal{D}x\, e^{-S_\text{E}[x]}\delta(\tau_b[x])}{\int_{x(-\mathcal{T})=a}^{x(\mathcal{T})=a}\mathcal{D}x\, e^{-S_\text{E}[x]}}\right)_{\mathcal{T}\to iT}$$

- Non-perturbative definition of the decay rate
- Does not require analytic continuing potential
- Does not require saddle-point approximation

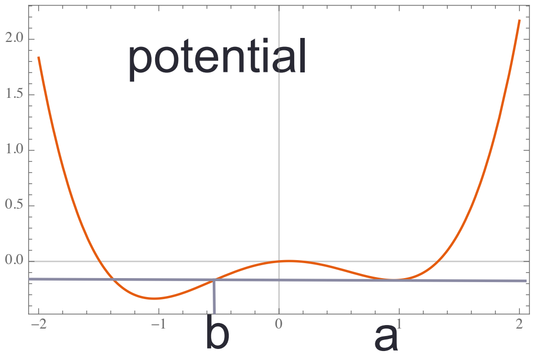
Expansion

Bounce dominates numerator

$$\Gamma_R = 2\mathrm{Im}\lim_{T\to\infty}\left(\frac{\int_{x(-\mathcal{T})=a}^{x(\mathcal{T})=a}\mathcal{D}x\, e^{-S_{\rm E}[x]}\delta(\tau_b[x])}{\int_{x(-\mathcal{T})=a}^{x(\mathcal{T})=a}\mathcal{D}x\, e^{-S_{\rm E}[x]}}\right)_{\mathcal{T}\to iT}$$

FV dominates denominator

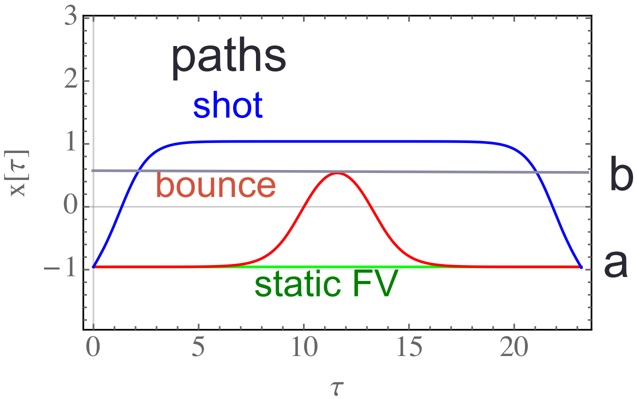
$$\approx \frac{\exp(-S_{\text{shot'}}) + \exp(-S_{\text{bounce}})}{\exp(-S_{\text{shot}}) + \exp(-S_{\text{bounce}}) + \exp(-S_{\text{FV}})}$$



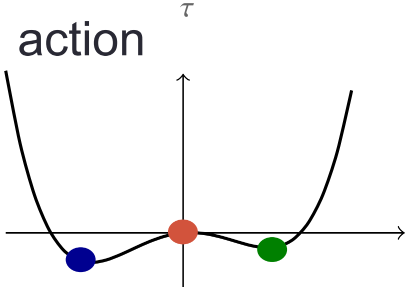
$$S_{\text{shot}} = E_{\text{TV}}\mathcal{T} + S_S^0 = iE_{\text{TV}}T + S_S^0$$
$$S_{\text{FV}} = E_{\text{FV}}\mathcal{T} = iE_{\text{FV}}T$$
$$S_{\text{bounce}} = E_{\text{FV}}\mathcal{T} + S_B^0 = iE_{\text{FV}}T + S_B^0$$

$$S_S^0 > S_B^0 > 0$$

=2.5 =1.2



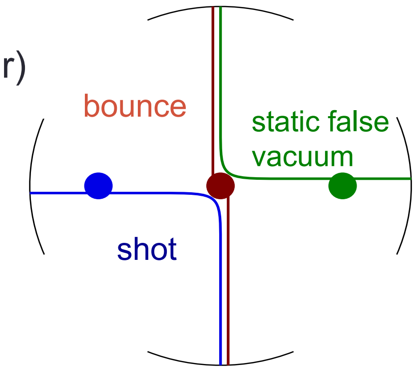
shot must go faster than bounce,
→ it has more kinetic energy



$$e^{-S_{\text{FV}}} \gg e^{-S_{\text{bounce}}} \gg e^{-S_{\text{shot}}}$$

Factor of 1/2

Γ= Im (FV contour) = ½ Im (bounce contour)



forces all paths to hit b at time t=0

$$\Gamma_R = 2\mathrm{Im}\lim_{\mathcal{T}\to\infty}\left(\frac{\int_{x(-\mathcal{T})=a}^{x(\mathcal{T})=a}\mathcal{D}x\, e^{-S_{\mathrm{E}}[x]}\delta(\tau_b[x])}{\int_{x(-\mathcal{T})=a}^{x(\mathcal{T})=a}\mathcal{D}x\, e^{-S_{\mathrm{E}}[x]}}\right)_{\mathcal{T}\to i T}$$

Expand around bounce: $x(\tau) = \bar{x}(\tau) + \sum \xi_n y_n(\tau)$

Hits b at its maximum

linear in ξ_n

$$\Gamma^{\mathrm{NLO}} = \frac{e^{-S_E[\bar{x}]}}{e^{-S_E[x_{FV}]}}\lim_{\mathcal{T}\to\infty}\left|\frac{2\mathrm{Im}\int d^n\zeta\ J[\tau_*(\zeta),\zeta]e^{-\frac{1}{2}\sum\lambda_i\zeta_i^2}}{\int \mathcal{D}\delta x\ e^{-\frac{1}{2}S_E''[x_{FV}]\delta x^2}}\Theta[\xi_n y_n(0)]\right|$$

Must hit b

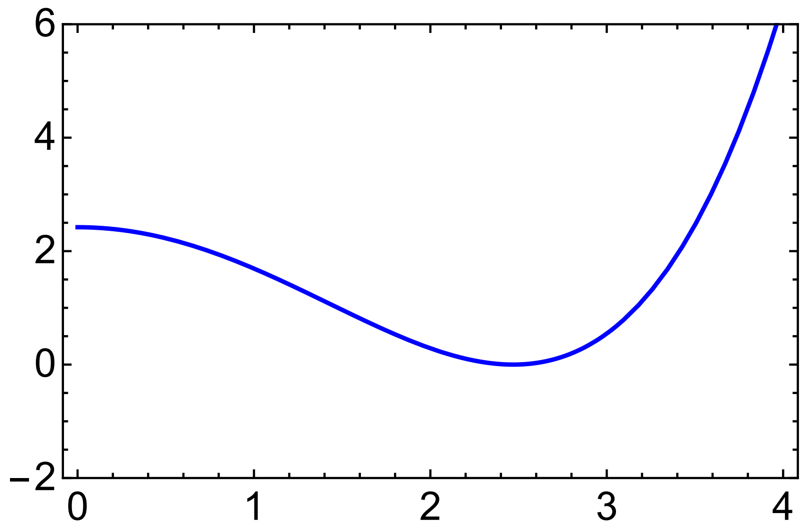
- Half the fluctuations don't hit b, half do
- Gaussian integral is symmetric.
- Can remove θ-function restriction and multiply by ½

$$\Gamma^{\mathrm{NLO}} = \frac{e^{-S_E[\bar{x}]}}{e^{-S_E[x_{FV}]}}\sqrt{\frac{S_E[\bar{x}]}{2\pi}}\left|\frac{\det'(-\partial_t^2 + V''(\bar{x}(t)))}{\det(-\partial_t^2 + V''(a))}\right|^{-1/2}$$

- Agrees with Coleman-Callan formula at NLO, but valid to all orders
- Does not rely on saddle-point approximation
- Actually connects formula to decay rate

EFFECTIVE POTENTIALS

How do we compute V_{eff}?



Classical potential: $V(h) = \Lambda + m^2 h^2 + \lambda h^4$

- Renormalizable
- Three parameters (Λ, m and λ), measured from data

How can the quantum-corrected potential be computed?

How do we compute V_{eff}?

Method 1:

$$\int \mathcal{D}H e^{i\Gamma} \equiv \int \mathcal{D}H\, \underbrace{\mathcal{D}\psi \cdots \mathcal{D}A e^{iS}}$$

Classical action

Effective Action

Integrate out everything but H

$$\Gamma = \int d^4x \Big\{ - Z[H] H \Box H - V_{\text{eff}}(H) + \cdots \Big\}$$

Problems:

- Generally non-local (has nasty things like $\ln \frac{1+\Box/m_t^2}{H^2}$ in it)
- Nearly impossible to compute

OK if $H \approx \langle H \rangle$

- Can't include loops of H itself this way

If we integrate over everything, effective action is just a number

$$e^{i\Gamma} = \int \mathcal{D}H \cdots \mathcal{D}A e^{iS}$$

Method 2: Legendre transform

Classical action

$$\frac{\delta S}{\delta H}\Big|_{H=v} = 0$$

Classical minimum

We want an effective action

$$\frac{\delta \Gamma}{\delta H}\Big|_{H=H_q} = 0$$

True quantum minimum

1. Compute W[J] $$e^{W[J]} \equiv \int \mathcal{D}H \cdots \mathcal{D}A e^{i\int d^4x\{\mathcal{L}+JH\}}$$

Current introduced by hand
So that Γ depends on something

2. Solve $H = \frac{\partial W}{\partial J}$ for J[H]

3. Compute $\Gamma[H] = W[J[H]] - \int d^4x H J[H]$

Has the property that $\frac{\delta \Gamma}{\delta H} = J[H]$ so that $\frac{\delta \Gamma}{\delta H} = 0$ when J=0 (i.e. in original theory)

- Agrees with method 1 in perturbation theory

What do you get?

Tree-level (classical)

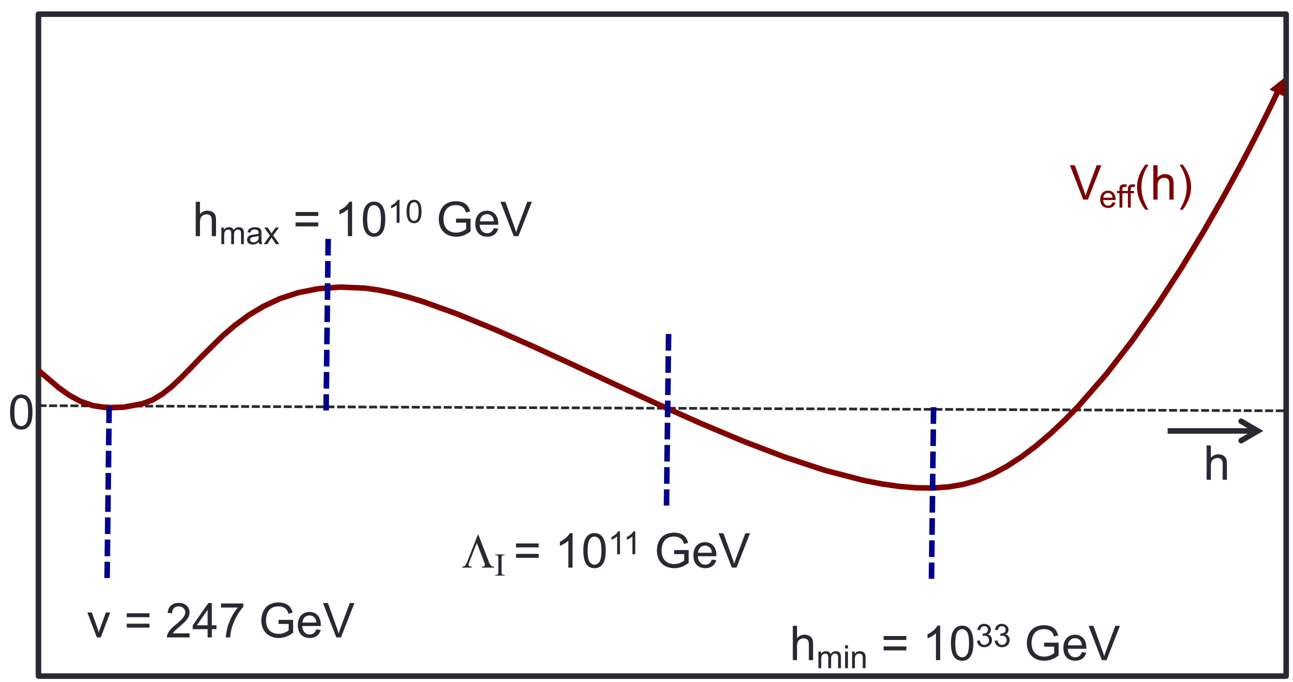
$$V_{\text{eff}} = \frac{1}{4}\lambda h^4 - m^2 h^2$$

$$+ h^4 \frac{1}{2048\pi^2}\left[-5g_1^4 + 6(g_1^2+g_2^2)^2 \ln\frac{h^2(g_1^2+g_2^2)}{4\mu^2} - 10 g_1^2 g_2^2 - 15 g_2^4 + 12 g_2^4 \ln\frac{g_2^2 h^2}{4\mu^2} + 144 y_t^4 - 96 y_t^4 \ln\frac{y_t^2 h^2}{2\mu^2}\right]$$

$$\frac{-1}{256\pi^2}\left[\xi_B g_1^2\left(\ln\frac{\lambda h^4(\xi_B g_1^2+\xi_W g_2^2)}{4\mu^4} - 3\right) + \xi_W g_2^2\left(\ln\frac{\lambda^3 h^{12}\xi_W^2 g_2^4(\xi_B g_1^2+\xi_W g_2^2)}{64\mu^{12}} - 9\right)\right]\lambda h^4$$

one-loop

$$+\cdots$$



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$$+\cdots$$

one-loop

Two curious features

1. Not gauge-invariant
2. Large logarithms

1. Gauge-dependence

Method 1 to compute Γ **is** gauge-invariant:

$$\int \mathcal{D}H e^{i\Gamma} \equiv \int \mathcal{D}H \underbrace{\mathcal{D}\psi \cdots \mathcal{D}A e^{iS}}$$

Completely integrate over gauge-orbits

Action/energy at minimum also gauge-invariant: $e^{i\Gamma} = \int \mathcal{D}H \cdots \mathcal{D}A e^{iS}$

Method 2 to compute Γ introduces a charged source J

$$e^{W[J]} \equiv \int \mathcal{D}H \cdots \mathcal{D}A e^{i\int d^4x \{\mathcal{L}+JH\}}$$

$$\Gamma = W - HJ$$

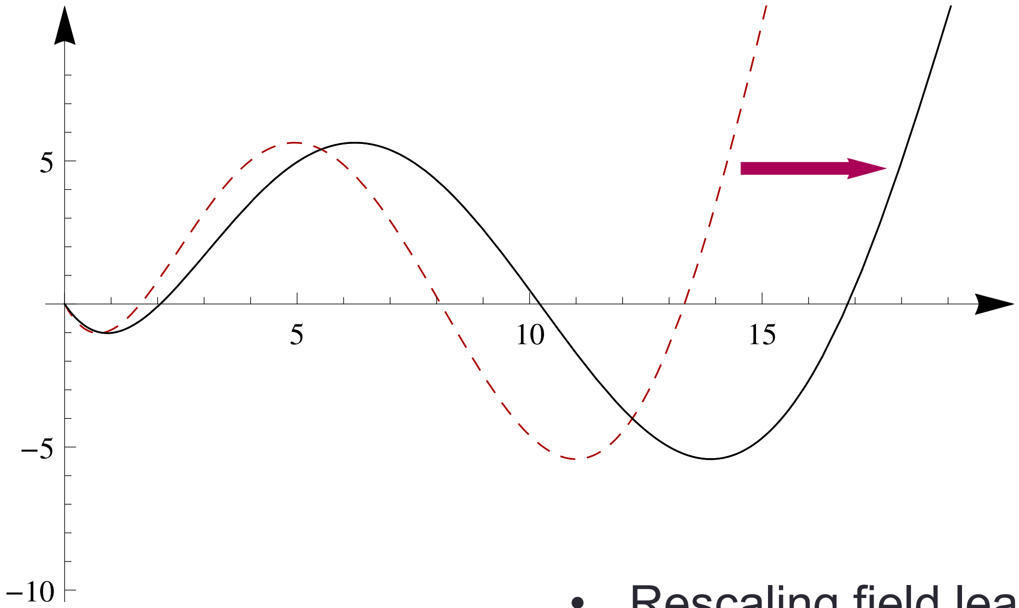
$$\frac{\delta\Gamma}{\delta H} = J$$

- Action away from minimum has current present
- Action **at minimum** has **no current**, should be gauge-invariant

Encoded in Nielsen identity

$$\left[\xi\frac{\partial}{\partial\xi} + C(h,\xi)\frac{\partial}{\partial h}\right] V_{\text{eff}}(h,\xi) = 0$$

Potential at minimum indep. of rescaling

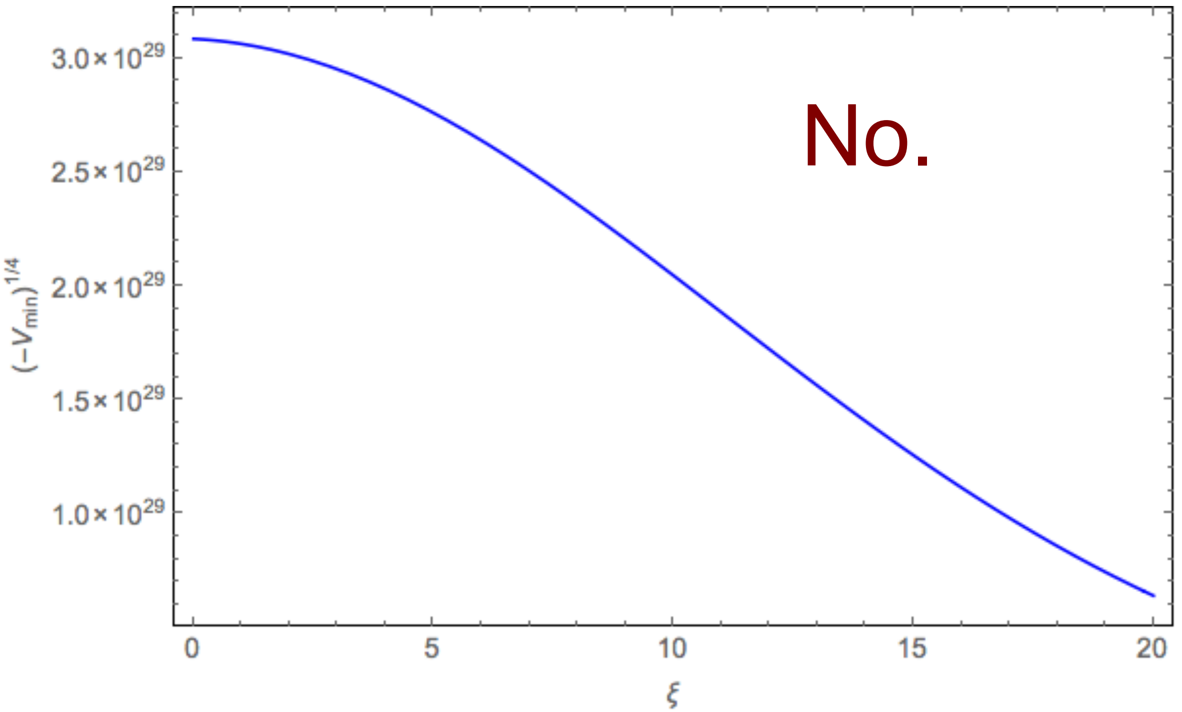


- Rescaling field leaves V_{min} unchanged

Nielsen identity

$$\left[\xi\frac{\partial}{\partial\xi}+C(h,\xi)\frac{\partial}{\partial h}\right]V_{\text{eff}}(h,\xi)=0$$

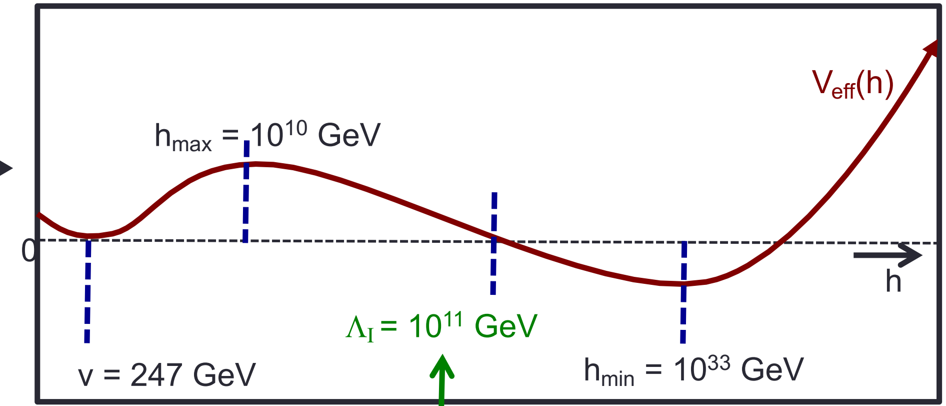
But is it?



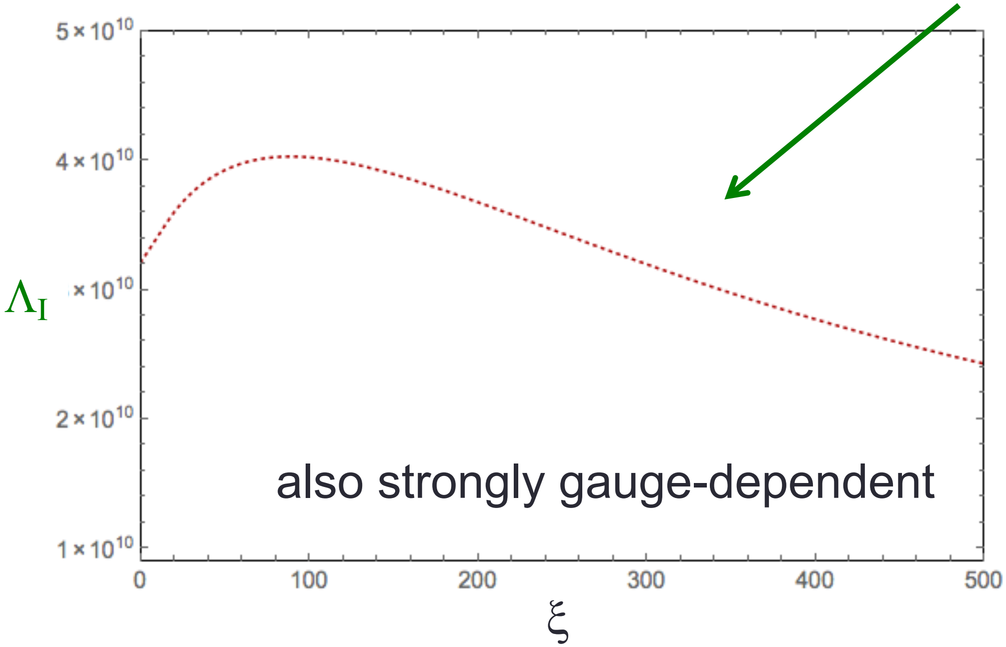
$(-V_{\min})^{1/4}$ appears linearly-dependent on gauge parameter ξ

What about field values?

Landau gauge (ξ=0) →



Instability scale Λ_I = value of h where V(h) = 0



also strongly gauge-dependent

- h_{min} also gauge dependent
- h_{max} also gauge dependent
- …

2. Large Logarithms

Can be resummed with RGE:

Explicit μ dependence

$$\left(\mu\frac{\partial}{\partial\mu}+\beta_i\frac{\partial}{\partial g_i}-\gamma h\frac{\partial}{\partial h}\right)V_{\text{eff}}=0$$

compensated for by rescaling couplings and fields

- Same RGE as 1PI Green's functions or off-shell matrix elements
- Observables/S-matrix elements satisfy simpler RGE:

$$\left(\mu\frac{\partial}{\mu}+\beta_i\frac{\partial}{\partial g_i}\right)\sigma=0$$

- Field-rescaling term canceled by LSZ wavefunction Z-factors

⇒ Effective potential depends on the normalization of fields??!!

Resum logarithms

1. Compute V_{eff} to fixed order (say 2-loops) at scale (say) μ_0 ~ 100 GeV
2. Solve RGE $\left(\mu\frac{\partial}{\partial\mu}+\beta_i\frac{\partial}{\partial g_i}-\gamma h\frac{\partial}{\partial h}\right)V_{\text{eff}}=0$

$$V_{\text{eff}}(h,g_i,\mu)\to V_{\text{eff}}(e^{\Gamma(\mu_0,\mu)}h,g_i(\mu),\mu)$$

$$\Gamma(\mu_0,\mu)\equiv\int_{\mu_0}^{\mu}\gamma(\mu')d\ln\mu'$$

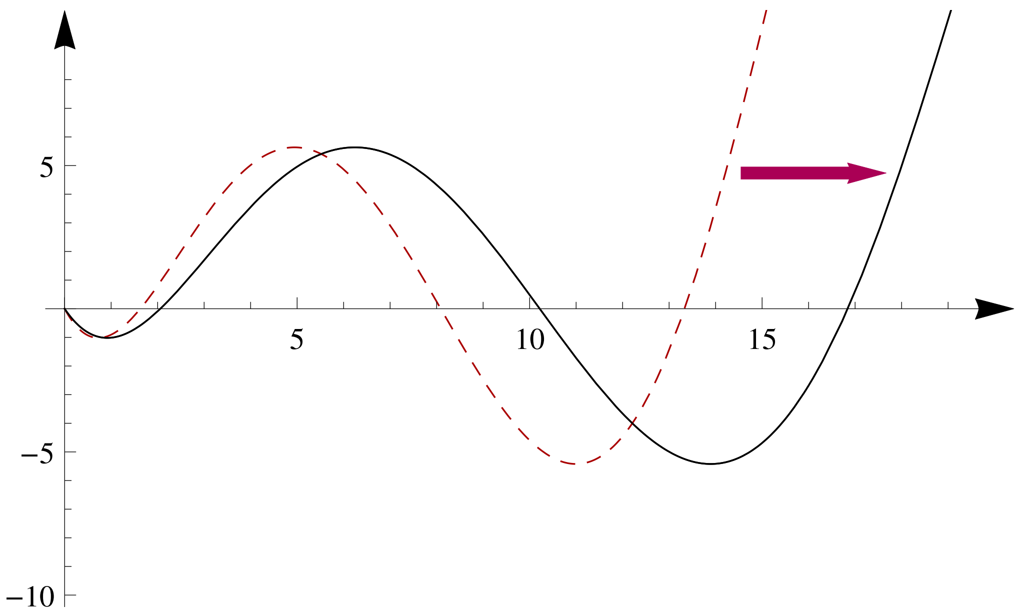
3. Set μ ~ h

$$V_{\text{eff}}(h,\mu_0)=V_{\text{eff}}(e^{\Gamma(\mu_0,h)}h,g_i(h),h)$$

Potential depends on scale μ_0 where it is calculated??!!

$$\Rightarrow\quad\left(\frac{\partial}{\partial\mu_0}-\gamma h\frac{\partial}{\partial h}\right)V(h,\mu_0)=0$$

Potential at minimum



Nielsen identity (gauge invariance)

$$\left[\xi\frac{\partial}{\partial\xi}+C(h,\xi)\frac{\partial}{\partial h}\right]V_{\text{eff}}(h,\xi)=0$$

Calculation-scale invariance

$$\left(\frac{\partial}{\partial\mu_0}-\gamma h\frac{\partial}{\partial h}\right)V(h,\mu_0)=0$$

V_{min} should be gauge invariant and independent of how it is calculated

Even gauge-invariant Γ is unphysical

Even if we source a gauge-invariant field $e^{W[J]} \equiv \int \mathcal{D}H \cdots \mathcal{D}A e^{i\int d^4x\{\mathcal{L}+JH\}}$

$$\left.\begin{aligned} e^{W[J]} &\equiv \int \mathcal{D}H \cdots \mathcal{D}A e^{i\int d^4x\{\mathcal{L}+JH^\dagger H\}} \\ e^{W[J]} &\equiv \int \mathcal{D}H \cdots \mathcal{D}A e^{i\int d^4x\{\mathcal{L}+J|H|\}} \end{aligned}\right\}$$

Γ(h) is now gauge-invariant

Effective potential still depends on how it is calculated $\left(\frac{\partial}{\partial\mu_0} - \gamma h \frac{\partial}{\partial h}\right) V(h,\mu_0) = 0$

- This is OK.
- Off-shell quantities can be unphysical

- **Observables should be physical**
  - S-matrix elements
  - Vacuum energy (V_{min})
  - Tunnelling rates
  - Critical temperature

But are they?

What about field values?

Instability scale?
Inflation scale?
Planck/new physics sensitivity?

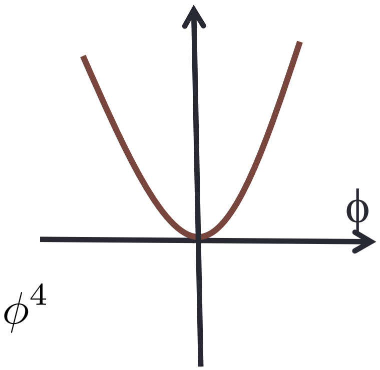
Are these questions about observables?

SCALAR QED

Scalar QED

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}|D_\mu\phi|^2 - V(\phi)$$

$$V_0(\phi) = \frac{\lambda}{24}\phi^4$$



- mass term gives small corrections, so we drop it

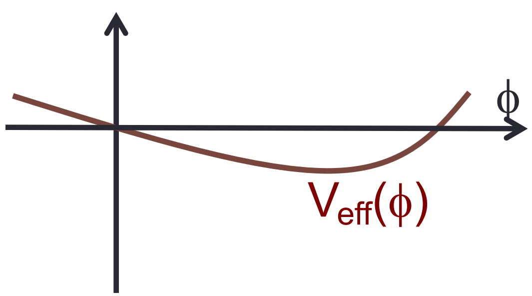
1-loop potential in R_ξ gauges:

$$V_1(\phi) = \phi^4 \frac{\hbar}{16\pi^2}\left[\frac{3}{4}e^4\left(\ln\frac{e^2\phi^2}{\mu^2} - \frac{5}{6}\right) + \frac{\lambda^2}{16}\left(\ln\frac{\lambda\phi^2}{2\mu^2} - \frac{3}{2}\right) + \left(\frac{\lambda^2}{144} - \frac{1}{12}e^2\lambda\xi\right)\left(\ln\frac{\phi^2}{\mu^2} - \frac{3}{2}\right) + \frac{1}{4}K_+^4 \ln K_+^2 + \frac{1}{4}K_-^4 \ln K_-^2\right]$$

$$K_\pm^2 = \frac{1}{12}\left(\lambda \pm \sqrt{\lambda^2 - 24\lambda e^2\xi}\right)$$

- Not gauge-invariant

- For most values of e and λ, there is no minimum
- When $\lambda \approx \frac{e^4}{16\pi^2}$ $\Rightarrow V_0 \approx V_1$
- And…. V_{min} depends on ξ



Spontaneous symmetry breaking

When is $\lambda \approx \frac{e^4}{16\pi^2}$?

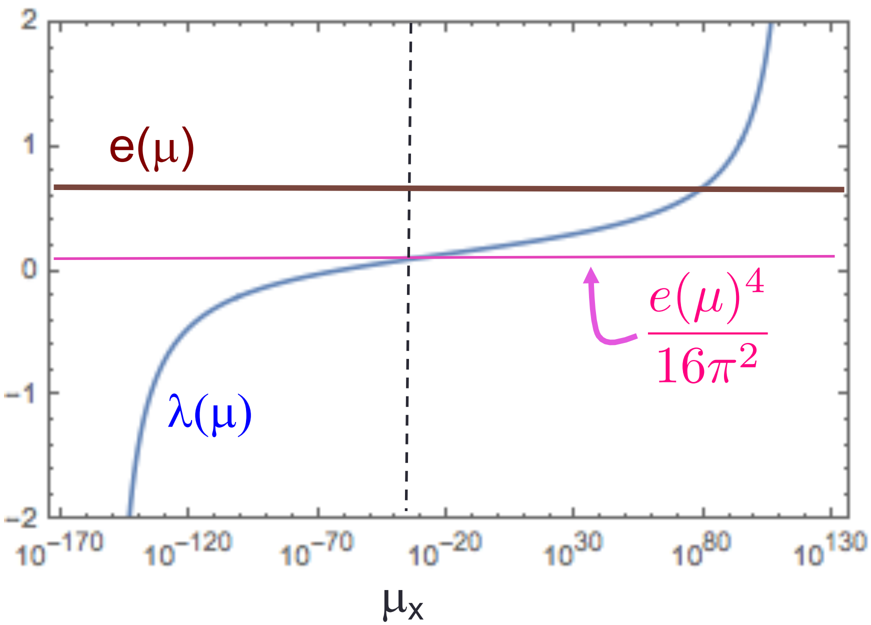
Solve RGEs:

$$\beta_e = \frac{\hbar}{16\pi^2}\left(\frac{e^3}{3}\right) + \cdots$$

$$\beta_\lambda = \frac{\hbar}{16\pi^2}\left(36e^4 - 12e^2\lambda + \frac{10\lambda^2}{3}\right)$$

$$e^2(\mu) = \frac{e^2(\mu_0)}{1 - \frac{e^2(\mu_0)}{24\pi^2}\ln\frac{\mu}{\mu_0}}$$

$$\lambda(\mu) = \frac{e^2(\mu)}{10}\left[19 + \sqrt{719}\tan\left(\frac{\sqrt{719}}{2}\ln\frac{e(\mu)^2}{C}\right)\right]$$



- e runs relatively slowly
- For any e, λ runs through all values
- There is always a scale μ_X where

$$\lambda(\mu_X) \approx \frac{e(\mu_X)^4}{16\pi^2}$$

- Near this scale, V_{eff} is pertubative

Proper loop expansion

$$V_0(\phi) = \frac{\lambda}{24}\phi^4$$

$$V_1(\phi) = \phi^4 \frac{\hbar}{16\pi^2}\left[\frac{3}{4}e^4\left(\ln\frac{e^2\phi^2}{\mu^2} - \frac{5}{6}\right) + \frac{\lambda^2}{16}\left(\ln\frac{\lambda\phi^2}{2\mu^2} - \frac{3}{2}\right) + \left(\frac{\lambda^2}{144} - \frac{1}{12}e^2\lambda\xi\right)\left(\ln\frac{\phi^2}{\mu^2} - \frac{3}{2}\right) + \frac{1}{4}K_+^4 \ln K_+^2 + \frac{1}{4}K_-^4 \ln K_-^2\right]$$

$$K_\pm^2 = \frac{1}{12}\left(\lambda \pm \sqrt{\lambda^2 - 24\lambda e^2\xi}\right)$$

Comparable when

$$\lambda \approx \hbar \frac{e^4}{16\pi^2}$$

- Then V0 and V1 of order $\hbar$

These terms all have extra $\hbar$ suppression

Expanding in $\hbar$ with $\lambda \sim \hbar$

order $\hbar$:

$$V^{\rm LO} = \frac{\lambda}{24}\phi^4 + \frac{\hbar e^4}{16\pi^2}\phi^4\left(-\frac{5}{8} + \frac{3}{2}\ln\frac{e\phi}{\mu}\right) \longrightarrow V_{\rm min}^{\rm LO} = -\frac{3}{128\pi^2}e^4\langle\phi\rangle^4$$

order $\hbar^2$:

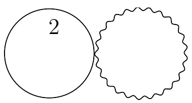
$$V^{\rm NLO} = \frac{\hbar e^2\lambda}{16\pi^2}\phi^4\left(\frac{\xi}{8} - \frac{\xi}{24}\ln\frac{e^2\lambda\xi\phi^4}{6\mu^4}\right) \longrightarrow V_{\rm min}^{\rm NLO} = \cdots$$

Problem: higher-loop contributions also of order $\hbar^2$

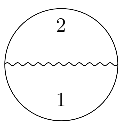
2-Loop potential in scalar QED

- Known in Landau gauge
- Some terms computed by Kang (1974), not in $\overline{\text{MS}}$
- Some terms at order $e^6\hbar^2$ unknown

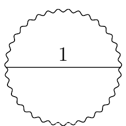
We computed all the relevant 2-loop graphs:



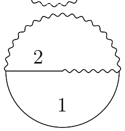
$$= \frac{\hbar^2\phi^4 e^6}{(16\pi^2)^2}\xi\left[-12\ln^2\frac{e\phi}{\mu}+\left(8-3\ln\frac{\lambda\xi}{6e^2}\right)\ln\frac{e\phi}{\mu}-\frac{5}{2}-\frac{\pi^2}{16}-\frac{3}{16}\ln^2\frac{\lambda\xi}{6e^2}+\ln\frac{\lambda\xi}{6e^2}\right]$$



$$= \frac{\hbar^2\phi^4 e^6}{(16\pi^2)^2}\left[(2+6\xi)\ln^2\frac{e\phi}{\mu}-(3+7\xi)\ln\frac{e\phi}{\mu}+\frac{7}{4}+\frac{\pi^2}{8}+\frac{15}{4}\xi+\frac{3\pi^2}{8}\xi\right]$$



$$= \frac{\hbar^2\phi^4 e^6}{(16\pi^2)^2}\left[(18+6\xi)\ln^2\frac{e\phi}{\mu}-(21+7\xi)\ln\frac{e\phi}{\mu}+\frac{47}{4}+\frac{7\pi^2}{24}+\frac{15}{4}\xi+\frac{3\pi^2}{8}\xi\right]$$



$$= \frac{\hbar^2\phi^4 e^6}{(16\pi^2)^2}\xi\left[-12\ln^2\frac{e\phi}{\mu}+14\ln\frac{e\phi}{\mu}-\frac{15}{2}-\frac{3\pi^2}{4}\right]$$

Then the relevant part of the 2-loop potential is

$$V_2=\left(\frac{\hbar}{16\pi^2}\right)^2 e^6\phi^4\left[(10-6\xi)\ln^2\frac{e\phi}{\mu}+\left(-\frac{62}{3}+4\xi-\frac{3}{2}\xi\ln\frac{\lambda\xi}{6e^2}\right)\ln\frac{e\phi}{\mu}\right.$$
$$\left.+\xi\left(-\frac{1}{2}+\frac{1}{4}\ln\frac{\lambda\xi}{6e^2}\right)+\frac{71}{6}\right]+\cdots$$ terms of order $\hbar^3$

Potential at minimum

$$V^{\rm LO} = \frac{\lambda}{24}\phi^4 + \frac{\hbar e^4}{16\pi^2}\phi^4\left(-\frac{5}{8}+\frac{3}{2}\ln\frac{e\phi}{\mu}\right) \qquad V^{\rm NLO} = \frac{\hbar e^2\lambda}{16\pi^2}\phi^4\left(\frac{\xi}{8}-\frac{\xi}{24}\ln\frac{e^2\lambda\xi\phi^4}{6\mu^4}\right)$$

$$+\frac{\hbar^2 e^6}{(16\pi^2)^2}\phi^4\left[(10-6\xi)\ln^2\frac{e\phi}{\mu}+\left(-\frac{62}{3}+4\xi-\frac{3}{2}\xi\ln\frac{\lambda\xi}{6e^2}\right)\ln\frac{e\phi}{\mu}+\xi\left(-\frac{1}{2}+\frac{1}{4}\ln\frac{\lambda\xi}{6e^2}\right)+\frac{71}{6}\right]$$

- Solve V'(ϕ=v) =0 for λ(v):

$$\lambda = \frac{\hbar e^4}{16\pi^2}\left(6-36\ln\frac{ev}{\mu}\right)+\frac{\hbar e^6}{(16\pi^2)^2}\left\{-160-24\xi+(376+90\xi)\ln\frac{ev}{\mu}-240\ln^2\frac{ev}{\mu}+9\xi\ln\left[\frac{\xi\hbar\mu^2}{16\pi^2 v^2}\left(1-6\ln\frac{ev}{\mu}\right)\right]\right\}$$

- Plug in to V(v):

$$V_{\rm min} = v^4\frac{e^4\hbar}{16\pi^2}\left(-\frac{3}{8}\right)+v^4\frac{e^6\hbar^2}{(16\pi^2)^2}\frac{1}{12}\left\{62-9\xi+(-60+18\xi)\ln\frac{ev}{\mu}+\frac{9}{2}\xi\ln\left[\frac{e^2\xi\hbar}{16\pi^2}\left(1-6\ln\frac{ev}{\mu}\right)\right]\right\}$$

Still gauge-dependent!

Problem : $v = \langle\phi\rangle$ is gauge-dependent

Express $V_{\rm min}$ in terms of only other dimensionful scale: μ

In terms of μ_X

Define μ_X by

$$\lambda(\mu_X) \equiv \frac{\hbar}{16\pi^2} e^4(\mu_X)\Big[6 - 36\ln[e(\mu_X)]\Big]$$

- Tree-level vev is $v=\mu_X$
- Exact (non-perturbative) definition of μ_X

Then, vev is:

$$v = \mu_X + \mu_X \frac{\hbar e^2}{16\pi^2}\left\{-\frac{40}{9} + \frac{94}{9}\ln e - \frac{20}{3}\ln^2 e - \frac{\xi}{2} + \frac{3}{2}\xi \ln e + \frac{\xi}{4}\ln\left[\frac{\xi\hbar}{16\pi^2}(1-6\ln e)\right] - \frac{1}{6}\xi + \xi \ln e\right\}.$$

- gauge-dependent vev is OK – not physical

Potential at minimum is:

$$V_{\min} = \frac{e^4\hbar}{16\pi^2}\mu_X^4\left(-\frac{3}{8}\right) + \frac{e^6\hbar}{(16\pi^2)^2}\mu_X^4\left(\frac{71}{6} - \frac{62}{3} + 10\ln^2 e\right) + \frac{e^6\hbar}{(16\pi^2)^2}\mu_X^4\left(\frac{\xi}{4} - \frac{3}{2}\xi\ln e\right)$$

- gauge-dependent vacuum energy is **not OK**

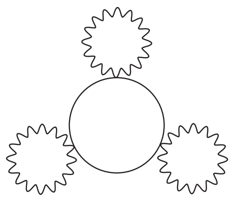
Still gauge-dependent!

What's missing?

More diagrams!

Daisy resummation

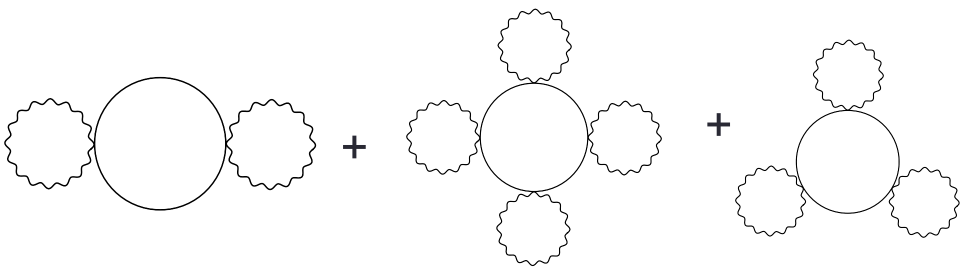
Higher order graphs can scale like inverse powers of λ:



$$\propto (e^2)^3(e^2\phi^2)^3 \int \frac{d^4k}{2\pi^4}\left(\frac{i}{k^2-\frac{\lambda}{2}\phi^2}\right)^3 \propto \phi^4\frac{e^{12}}{\lambda}$$

Effective masses depend on λ

Only one series of graphs contribute at order $\sim \hbar^2$



+ …

"daisy diagrams"

We can sum the series:

$$V^{e^6\text{daisies}} = \phi^4\frac{\hbar}{16\pi^2}\left(-\frac{e^2\lambda\xi}{24}\right)\left[\frac{\widehat{\lambda}(\phi)}{\lambda}+\left(1-\frac{\widehat{\lambda}(\phi)}{\lambda}\right)\ln\left(1-\frac{\widehat{\lambda}(\phi)}{\lambda}\right)\right]$$

$$\widehat{\lambda}(\phi) \equiv \frac{\hbar e^4}{16\pi^2}\left(6-36\ln\frac{e\phi}{\mu}\right)$$

Full potential at NLO:

$$V^{\rm NLO} = \frac{\hbar e^2 \lambda}{16\pi^2}\phi^4 \left(\frac{\xi}{8} - \frac{\xi}{24}\ln\frac{e^2\lambda\xi\phi^4}{6\mu^4}\right)$$
$$+\frac{\hbar^2 e^6}{(16\pi^2)^2}\phi^4\left[(10-6\xi)\ln^2\frac{e\phi}{\mu} + \left(-\frac{62}{3}+4\xi-\frac{3}{2}\xi\ln\frac{\lambda\xi}{6e^2}\right)\ln\frac{e\phi}{\mu} + \xi\left(-\frac{1}{2}+\frac{1}{4}\ln\frac{\lambda\xi}{6e^2}\right)+\frac{71}{6}\right]$$
$$+\phi^4\frac{\hbar e^2\lambda}{16\pi^2}\left(-\frac{\xi}{24}\right)\left[\frac{\widehat{\lambda}(\phi)}{\lambda} + \left(1-\frac{\widehat{\lambda}(\phi)}{\lambda}\right)\ln\left(1-\frac{\widehat{\lambda}(\phi)}{\lambda}\right)\right]$$

Now… vacuum energy is gauge-invariant!

$$V_{\rm min} = -\frac{3\hbar e^4}{128\pi^2}\mu_X^4 + \frac{e^6\hbar^2}{(16\pi^2)^2}\mu_X^4\left(\frac{71}{6} - \frac{62}{3}\ln e + 10\ln^2 e\right)$$

Field values are still gauge-dependent:

$$v = \mu_X + \mu_X\frac{\hbar e^2}{16\pi^2}\left\{-\frac{40}{9}+\frac{94}{9}\ln e - \frac{20}{3}\ln^2 e - \frac{\xi}{2} + \frac{3}{2}\xi\ln e + \frac{\xi}{4}\ln\left[\frac{\xi\hbar}{16\pi^2}(1-6\ln e)\right] - \frac{1}{6}\xi + \xi\ln e\right\}.$$

$$\Lambda_I = \mu_I + \mu_I\frac{\hbar e^2}{16\pi^2}\left\{-\frac{77}{9}+\frac{124}{9}\ln e - \frac{20}{3}\ln^2 e - \frac{\xi}{2} + \frac{3}{2}\xi\ln e + \frac{\xi}{4}\ln\left[\frac{\xi\hbar}{16\pi^2}(1-6\ln e)\right] - \frac{5}{12}\xi + \xi\ln e\right\}.$$

STANDARD MODEL

Lessons from scalar QED

1. Gauge invariance requires consistent expansion in $\hbar$

To N^nLO order

↙ Drop some n-loop contributions

↘ Include contributions from > n loops

2. Don't resum logs by solving RGE for V_{eff}

$$\left(\mu\frac{\partial}{\partial\mu}+\beta_i\frac{\partial}{\partial g_i}-\gamma h\frac{\partial}{\partial h}\right)V_{\text{eff}}=0$$

- Mixes up orders in $\hbar$ in an uncontrolled way

3. Do resum logs by using couplings at some scale μ_X

- Natural condition for μ_X is that $V_{LO}'(\phi=\mu_X) = 0$

4. Don't express V_{min} in terms of $v = \langle\phi\rangle$

- Express V_{min} in terms of μ_X instead

Standard Model

$$V^{(\mathrm{LO})}(h) = \frac{1}{4}\lambda h^4 + h^4 \frac{1}{2048\pi^2}\Big[-5g_1^4 + 6(g_1^2+g_2^2)^2 \ln\frac{h^2(g_1^2+g_2^2)}{4\mu^2} - 10g_1^2g_2^2 - 15g_2^4 + 12g_2^4 \ln\frac{g_2^2h^2}{4\mu^2} + 144y_t^4 - 96y_t^4 \ln\frac{y_t^2h^2}{2\mu^2}\Big]$$

Tree-level

Part of 1-loop $\lambda \sim \mathcal{O}(\hbar)$

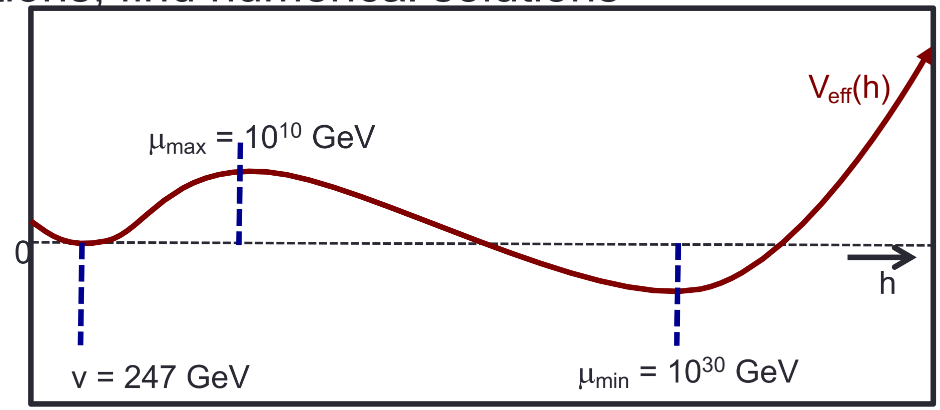
- Scale h=μ_X where $\frac{d}{dh}V^{(\mathrm{LO})}(h) = 0$ is

$$\lambda = \frac{1}{256\pi^2}\left[g_1^4 + 2g_1^2g_2^2 + 3g_2^4 - 48y_t^4 - 3\left(g_1^2+g_2^2\right)^2 \ln\frac{g_1^2+g_2^2}{4} - 6g_2^4 \ln\frac{g_2^2}{4} + 48y_t^4 \ln\frac{y_t^2}{2}\right]$$

- Run couplings with 3-loop β-functions, find numerical solutions

$$\mu_X^{\max} = 2.46 \times 10^{10}\ \mathrm{GeV}$$

$$\mu_X^{\min} = 3.43 \times 10^{30}\ \mathrm{GeV}$$



Standard Model at NLO

- We know the 1-loop contribution to V_{NLO}

$$V^{(1,\text{NLO})}(h) = \frac{-1}{256\pi^2}\left[\xi_B g_1^2\left(\ln\frac{\lambda h^4(\xi_B g_1^2+\xi_W g_2^2)}{4\mu^4} - 3\right) + \xi_W g_2^2\left(\ln\frac{\lambda^3 h^{12}\xi_W^2 g_2^4(\xi_B g_1^2+\xi_W g_2^2)}{64\mu^{12}} - 9\right)\right]\lambda h^4$$

- We know the 2-loop contribution to V_{NLO} in Landau gauge

$$\begin{aligned}
\lambda_{\text{eff}}^{(2)} = \frac{1}{(4\pi)^4}\Big[& 8g_3^2 y_t^4\left(3r_t^2-8r_t+9\right)+\frac{1}{2}y_t^6\left(-6r_t r_W-3r_t^2+48r_t-6r_{tW}-69-\pi^2\right)+\\
&+\frac{3y_t^2 g_2^4}{16}\left(8r_W+4r_Z-3r_t^2-6r_t r_Z-12r_t+12r_{tW}+15+2\pi^2\right)+\\
&+\frac{y_t^2 g_Y^4}{48}\left(27r_t^2-54r_t r_Z-68r_t-28r_Z+189\right)+\frac{y_t^2 g_2^2 g_Y^2}{8}(9r_t^2-18r_t r_Z+4r_t+44r_Z-57)+\\
&+\frac{g_2^6}{192}(36r_t r_Z+54r_t^2-414r_W r_Z+69r_W^2+1264r_W+156r_Z^2+632r_Z-144r_{tW}-2067+90\pi^2)+\\
&+\frac{g_2^4 g_Y^2}{192}(12r_t r_Z-6r_t^2-6r_W(53r_Z+50)+213r_W^2+4r_Z(57r_Z-91)+817+46\pi^2)+\\
&+\frac{g_2^2 g_Y^4}{576}(132r_t r_Z-66r_t^2+306r_W r_Z-153r_W^2-36r_W+924r_Z^2-4080r_Z+4359+\\
&+\frac{g_Y^6}{576}(6r_Z(34r_t+3r_W-470)-102r_t^2-9r_W^2+708r_Z^2+2883+206\pi^2)+\\
&+\frac{y_t^4}{6}\left(4g_Y^2(3r_t^2-8r_t+9)-9g_2^2\left(r_t-r_W+1\right)\right)+\frac{3}{4}\left(g_2^6-3g_2^4y_t^2+4y_t^6\right)\text{Li}_2\frac{g_2^2}{2y_t^2}+\\
&+\frac{y_t^2}{48}\xi(\frac{g_2^2+g_Y^2}{2y_t^2})\left(9g_2^4-6g_2^2g_Y^2+17g_Y^4+2y_t^2(7g_Y^2-73g_2^2+\frac{64g_2^4}{g_Y^2+g_2^2})\right)+\\
&+\frac{g_2^2}{64}\xi(\frac{g_2^2+g_Y^2}{g_2^2})\left(18g_2^2g_Y^2+g_Y^4-51g_2^4-\frac{48g_2^6}{g_Y^2+g_2^2}\right)\Big].
\end{aligned}$$

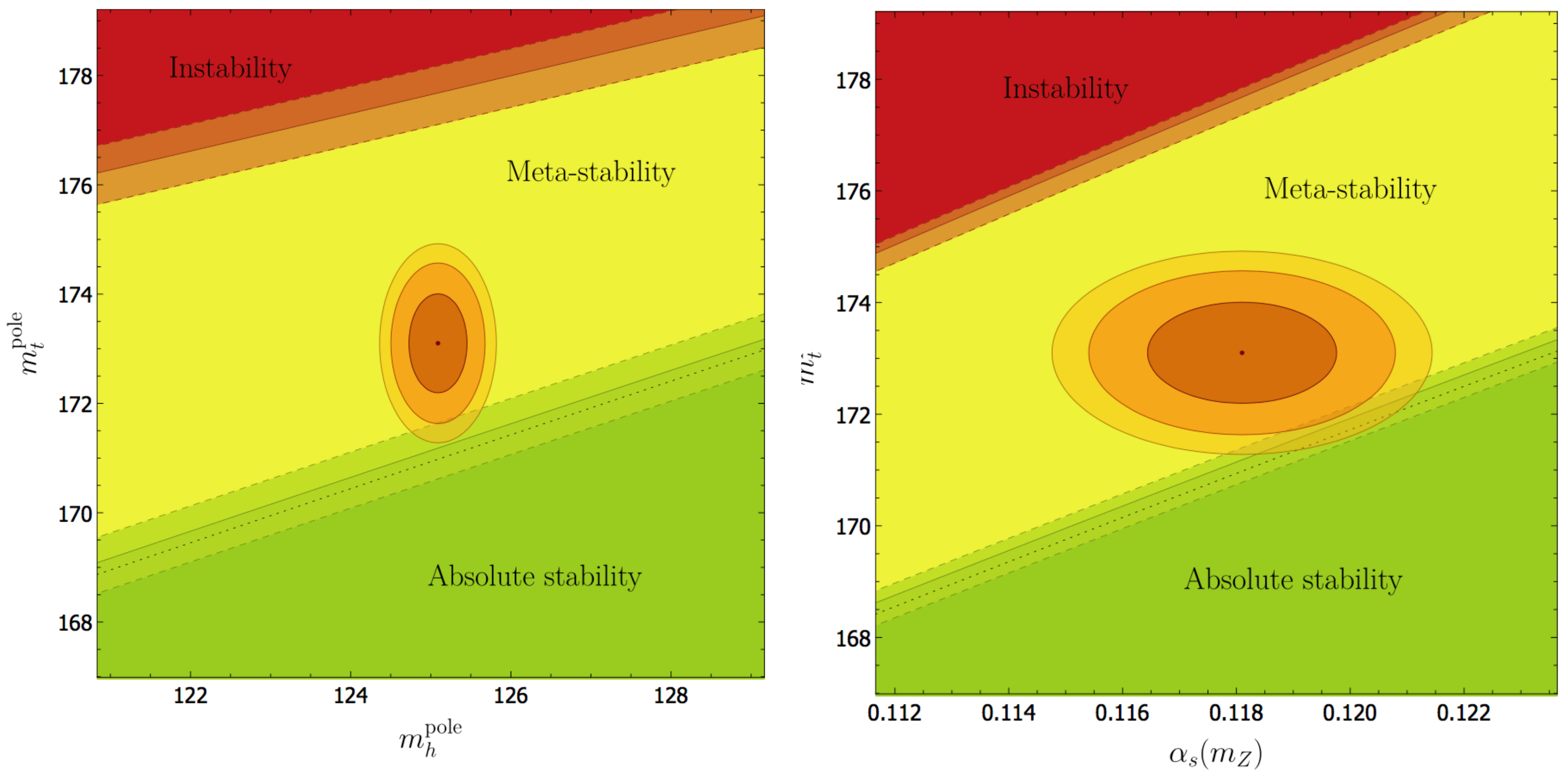
- We don't know the Daisy contribution. But we do know if vanishes in Landau gauge at NLO

$$V^{e^6\text{daisies}} = \phi^4\frac{\hbar}{16\pi^2}\left(-\frac{e^2\lambda\xi}{24}\right)\left[\frac{\widehat{\lambda}(\phi)}{\lambda}+(1-\frac{\widehat{\lambda}(\phi)}{\lambda})\ln(1-\frac{\widehat{\lambda}(\phi)}{\lambda})\right]$$

- Assuming everything works like in scalar QED, we have everything we need for NLO

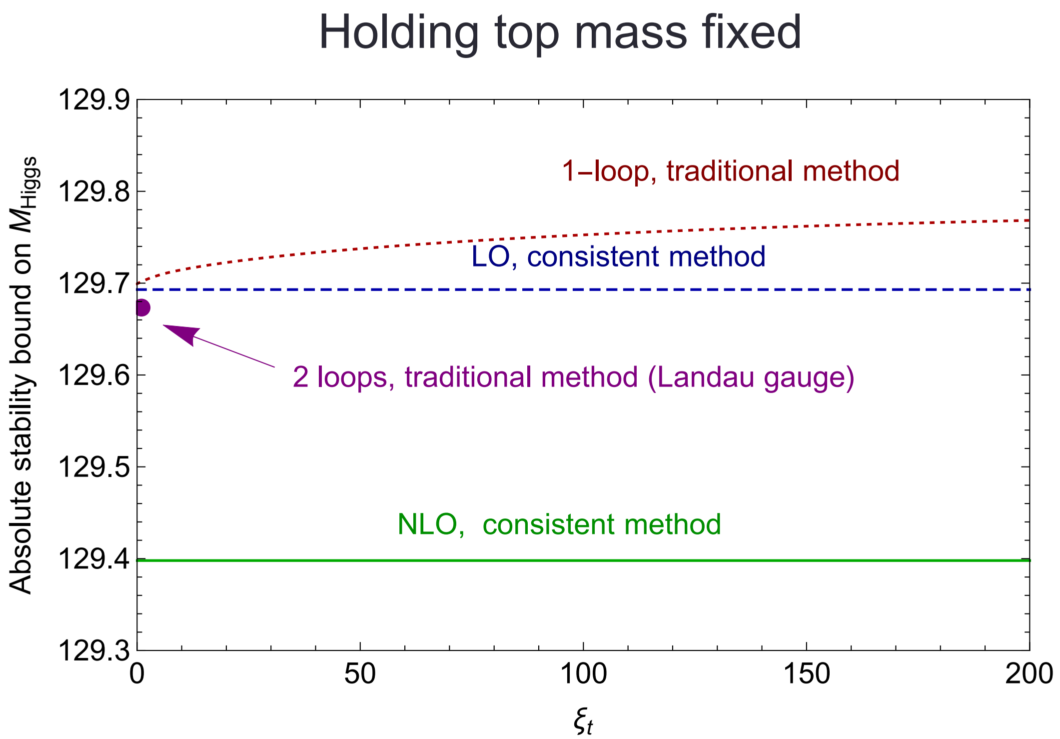
Results

Absolute stability: for what values of the Higgs and top masses is is $V_{min} = 0$?



Results

Absolute stability: for what values of the Higgs mass is $V_{min} = 0$ at fixed top mass?

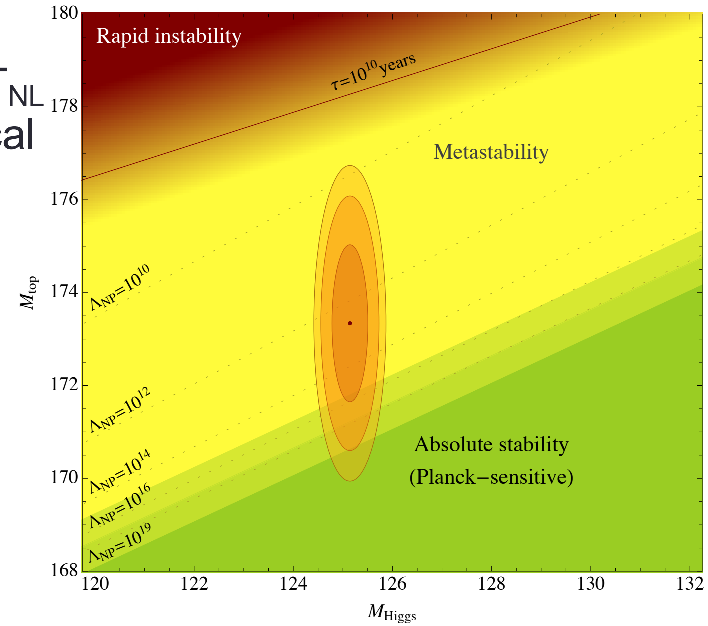


- Absolute stability bound lowered by 300 MeV
- Larger shift that including the 2-loop V_{eff}

Conclusions

Tunneling involves many exotic elements of quantum field theory

- Tunneling rates
  - Two time scales relevant for tunneling: $T_{slosh} << T << T_{NL}$
  - Asymptotic expansions and analytic continuation critical
    - Can be avoided with a direct approach
- Requires consistent use of perturbation theory
  - $\lambda \sim h$ power counting
- UV physics does not decouple
  - Stability is necessarily Planck-sensitive
  - Can make lifetime shorter, not longer



Do we know if the universe is stable?

- Our universe will probably decay, eventually.
- We don't know how long it will last

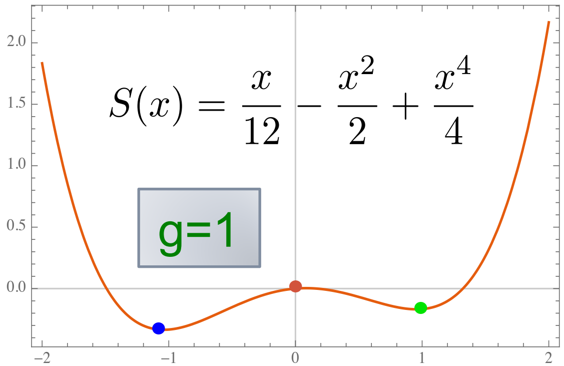
2. SCHRODINGER EQUATION

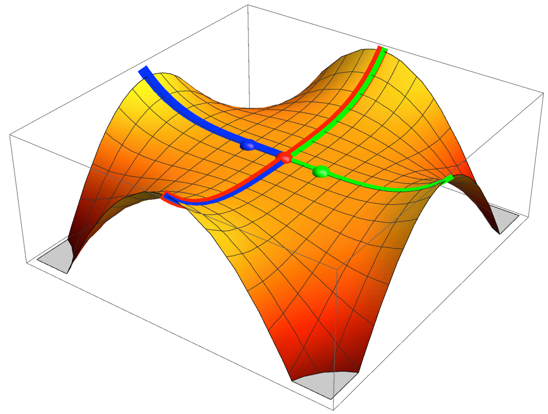
Gamow (1928) & Siegert (1939)

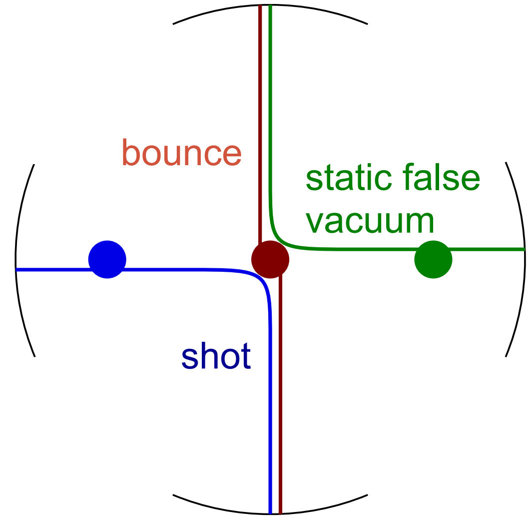
1. THE POTENTIAL DEFORMATION METHOD (CONTINUED)

Colelman & Callan (1977)

Potential deformation





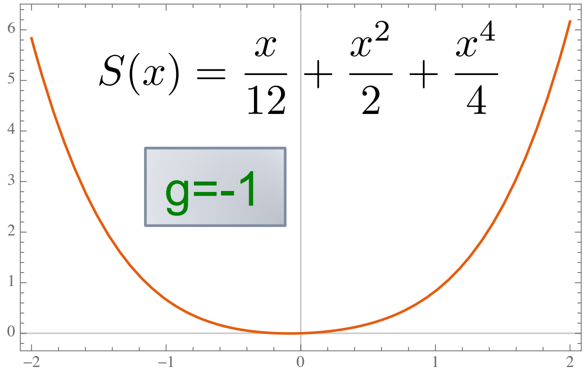


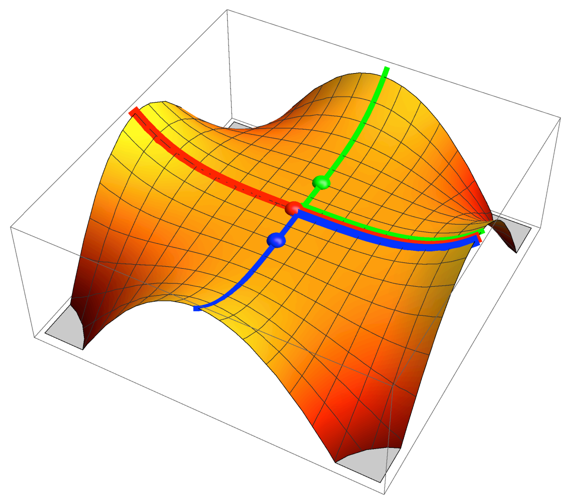
$$S_g(x) = \frac{x}{12} - g\frac{x^2}{2} + \frac{x^4}{4}$$

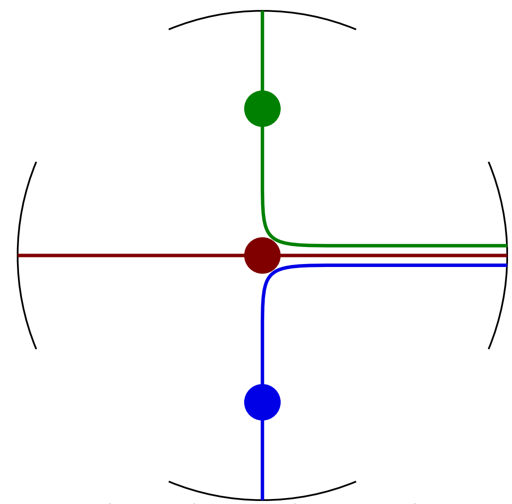
deform potential to prevent tunneling

$$Z_g = \int_{-\infty}^{\infty} dx e^{-S_g(x)}$$

- real at g = +1
- real at g = -1
- an analytic function of g



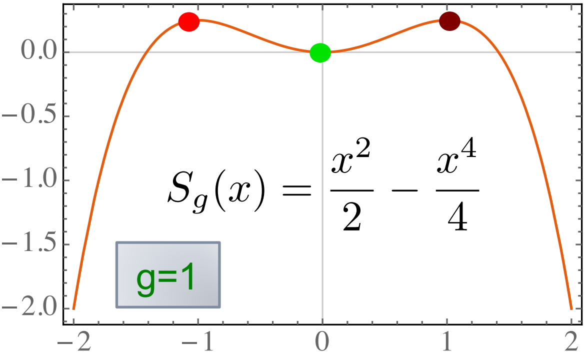


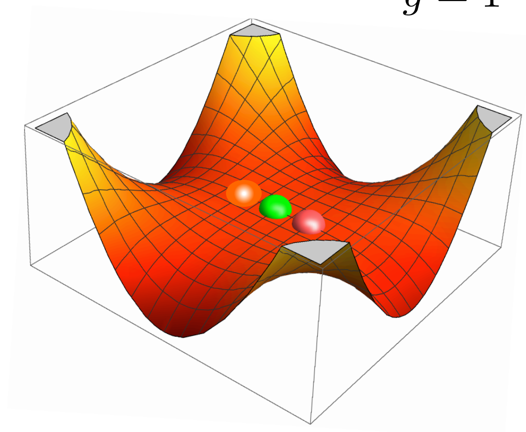


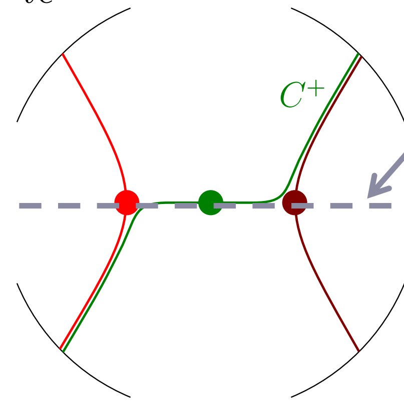
More standard example

$$Z_g = \int_{-\infty}^{\infty} dx e^{-S_g(x)}$$

$g = 1 + i\epsilon$



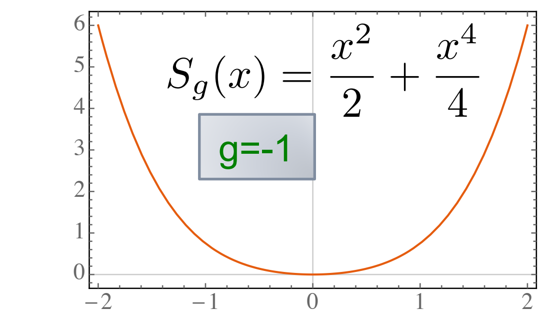


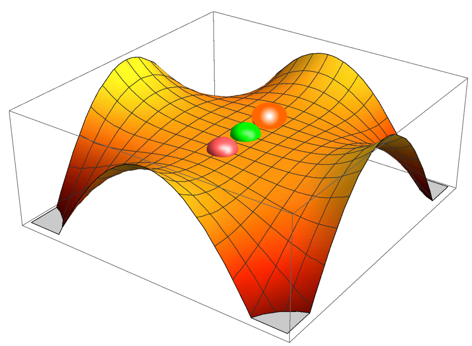


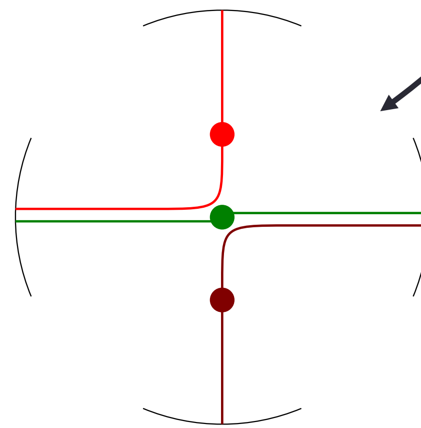
Divergent along physical integration contour (real axis)

$$S_g(x) = \frac{x^2}{2} - g\frac{x^4}{4}$$

Analytically continue to g=-1







- Fix integration to be along contour passing through saddle at x=0
- Return to g=1, keeping integration along green contour
- Z now has imaginary part at g=1

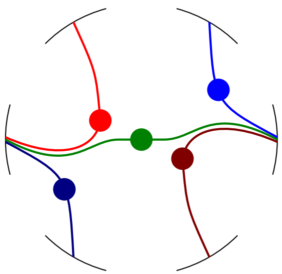
Well-defined procedure. But is the imaginary part the decay rate?

Add convergence factor

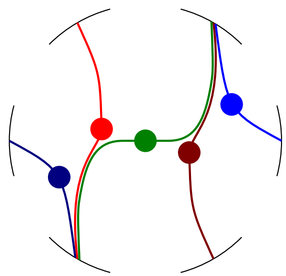
$$S_g(x) = \frac{x^2}{2} - g\frac{x^4}{4} + \frac{x^6}{60}$$

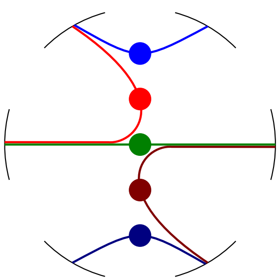
- Modifying potential/action away from region of interest should not affect rate

$g = \exp\left(i\frac{\pi}{4}\right)$

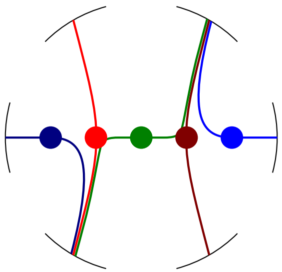


$g = \exp\left(i\frac{\pi}{3}\right)$

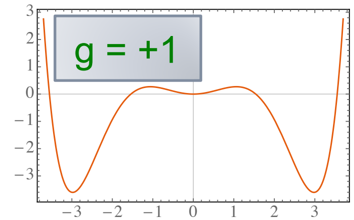


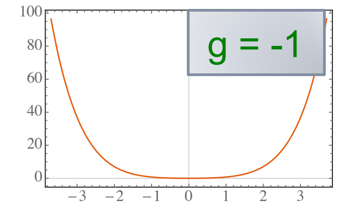


$g = -\exp\left(-i\epsilon\right)$



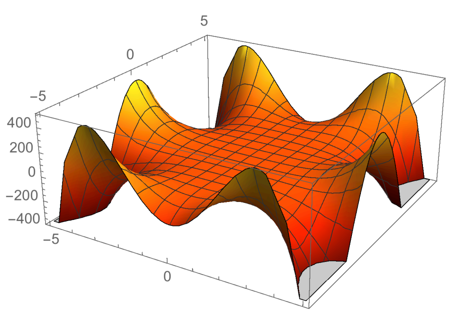
$g = \exp\left(i\epsilon\right)$

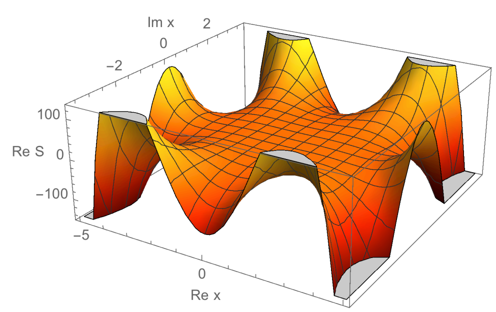




$$Z_g = \int_{-\infty}^{\infty} dx e^{-S_g(x)}$$

- real at g = +1
- real at g = -1
- Z_g is an analytic function of g





- We can still fix the contour at g=-1 and follow it back.

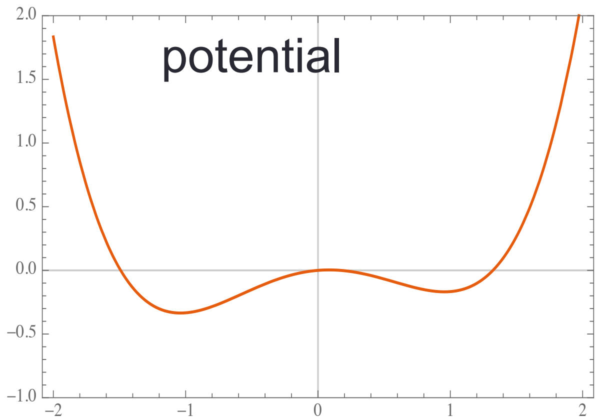
But why?

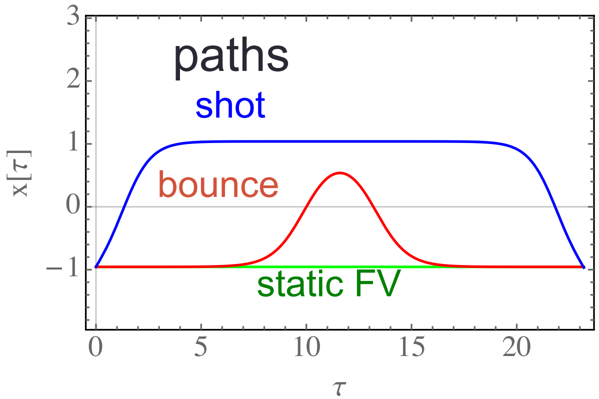
Physical limits

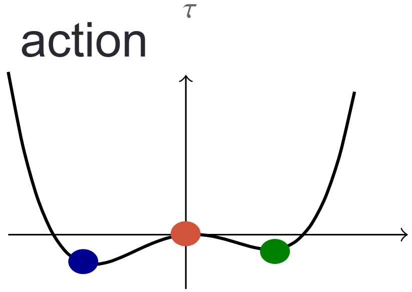
T >> T_{slosh} (only metastable FV decay)

T << T_{NL} (no return flux)

$$\Gamma = -\lim_{\frac{T}{T_{\text{slosh}}}\to\infty}\ \lim_{\frac{T}{T_{\text{NL}}}\to 0} \frac{1}{P_{\text{FV}}}\frac{d}{dT}P_{\text{FV}}$$







$$Z \equiv \langle a|e^{-H\mathcal{T}}|a\rangle = \int_{x(0)=a}^{x(\mathcal{T})=a} \mathcal{D}x e^{-S_E[x]}$$

$$\sim e^{-E_0 T} + e^{-E_{FV}T}$$

$$-\lim_{T\to\infty}\frac{1}{T}\ln Z = \min(E_0, E_{FV})$$

Taking $T \to \infty$ picks out true ground state E_0

Physical limits

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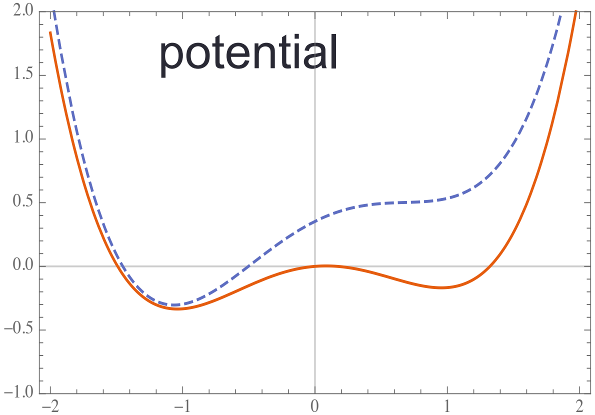
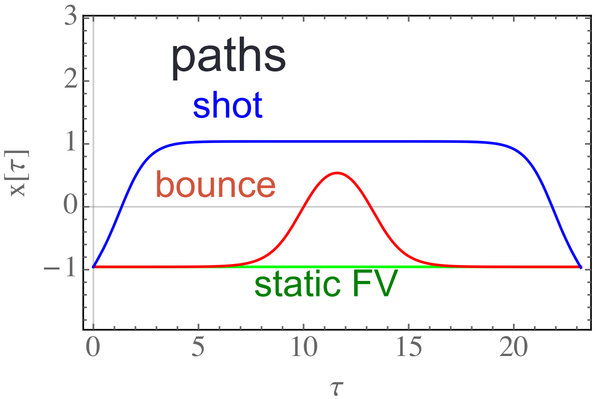


$$Z \equiv \langle a|e^{-H\mathcal{T}}|a\rangle = \int_{x(0)=a}^{x(\mathcal{T})=a}\mathcal{D}x e^{-S_E[x]}$$

$$\sim e^{-E_0 T} + e^{-E_{FV}T}$$

$$\Rightarrow -\lim_{T\to\infty}\frac{1}{T}\ln Z = \min(E_0, E_{FV})$$

Taking $T\to\infty$ picks out true ground state E_0



We want to

1. Deform the potential so FV is true ground state
2. Take $T\to\infty$
   - Picks out E_{FV}(g)
3. Deform back

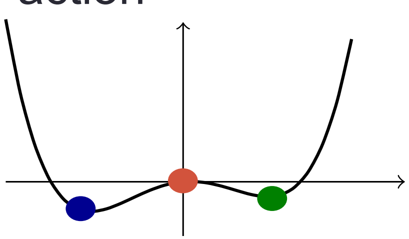
T << T_{NL} (no return flux)

T >> T_{slosh} (only metastable FV decay)

The $T\to\infty$ limit **does not commute** with analytic continuation

- $\min(E_0, E_{FV})$ is **not analytic**

action



$T\to\infty$ limit
like $\hbar\to 0$ limit
forces saddle point approximation

$$Z \sim \int dx e^{-S_E} \sim \int dx e^{-\frac{TE_{FV}}{\hbar}}$$

Saddle point approximation

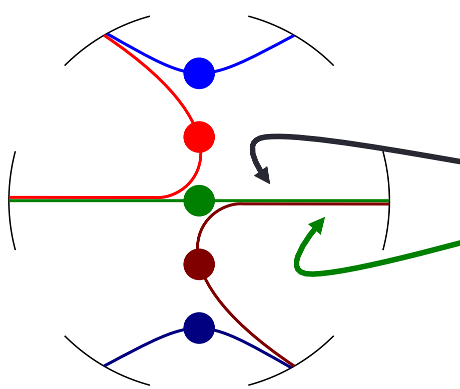
$$Z_g = \int dx e^{\frac{1}{\hbar}(-\frac{1}{2}x^2 + \frac{g}{4}x^4)}$$

$$\approx \int dx e^{-\frac{1}{2\hbar}x^2}\left[1 + \frac{g}{4}x^4 + \frac{1}{2}\left(\frac{g}{4}x^4\right)^2 + \cdots\right]$$

$$= \sqrt{2\pi}\left(1 + \frac{3g}{4} + \frac{105g^2}{32} + \frac{3465g^3}{128} + \frac{675675g^4}{2048} + \frac{43648605g^5}{8192} + \frac{7027425405g^6}{65536} + \cdots\right)$$

- Asymptotic series
- Coefficients grow factorially
- Summing the series does **not** reproduce the original function

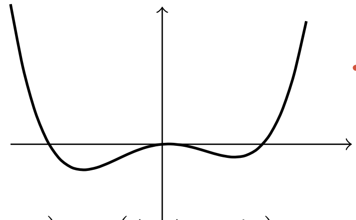
Performing the saddle point approximation does not commute with analytic continuation



move contour from real line to thimble

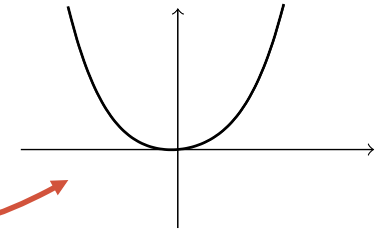
Examples

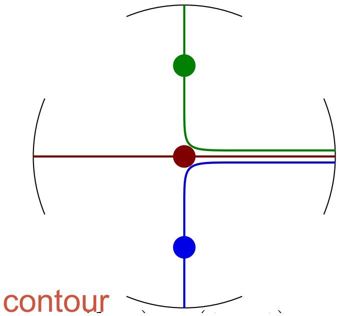
$$S_g(x) = h\frac{x}{12} - g\frac{x^2}{2} + \frac{x^4}{4}$$



$(h, g) = (1, 1 + i\epsilon)$

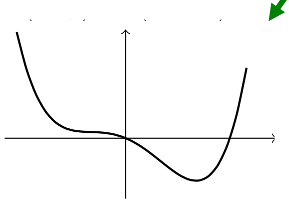
Deform to stabilize bounce



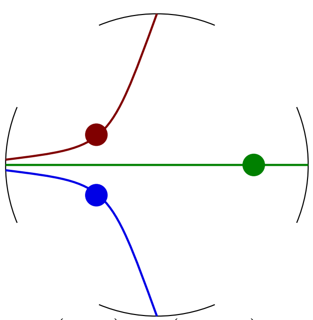


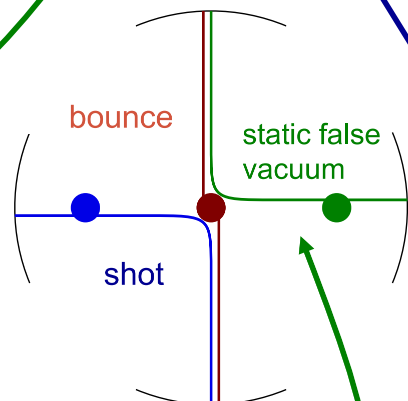
Z has imaginary part equal to **all** of the bounce contour

Deform to stabilize false vacuum



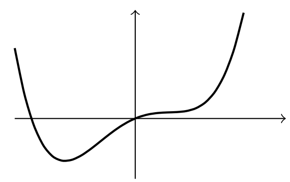
$(h, g) = (-5, 1)$

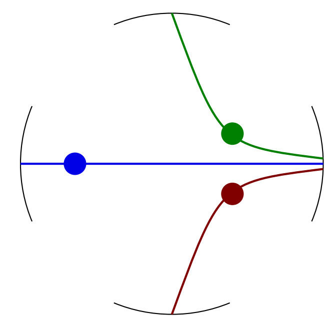




Z has imaginary part equal to **half** of the bounce contour

Deform to stabilize shot



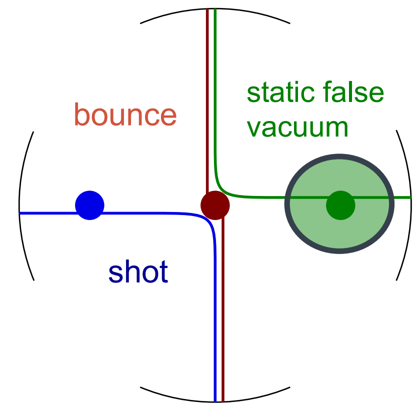


Z has imaginary part equal to **minus half** of the bounce contour

- Probability grows with time

T → ∞ limit fixes to green contour

Discontinuity



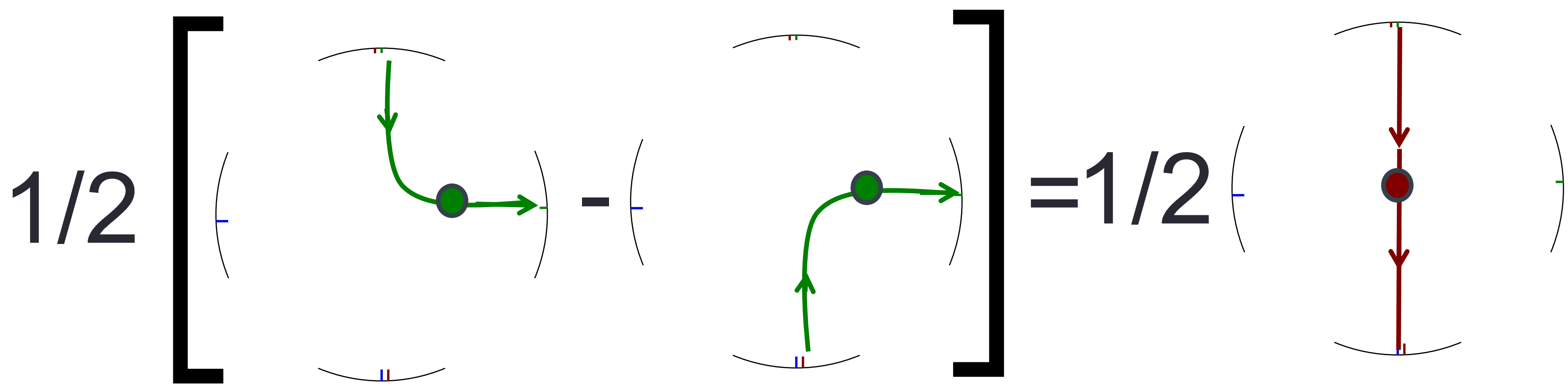
Can we just integrate along the FV contour?

Yes, at least for this toy integral

$$Z = \int_C dx e^{-S(x)}$$

No

- Not clear what "fixing to a contour" means for a path integral
- Saddle point approximation loses the imaginary part
  - Expanding around the saddle gives a real integral
  - Imaginary part comes from region far away
- Saddle point approximation does work for the discontinuity



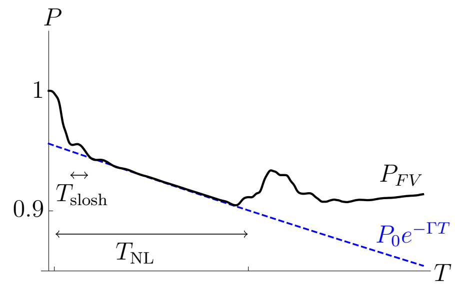
Summary of tunneling rates

T << T_{NL} (no return flux)

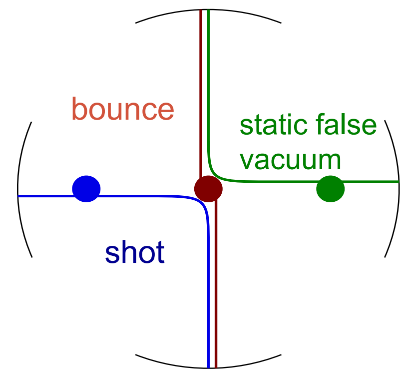
Precise definition of decay rate involves **two limits** $\Gamma = - \lim_{\frac{T}{T_{\text{slosh}}} \to \infty} \lim_{\frac{T}{T_{\text{NL}}} \to 0} \frac{1}{P_{\text{FV}}} \frac{d}{dT} P_{\text{FV}}$

T >> T_{slosh} (remove transients

Three methods to compute Γ

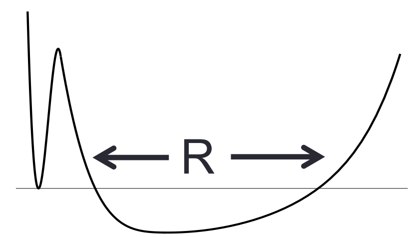


1
- **Solve Schrodingers equation**
- Impractical for QFT



2
- **Deform potential** to stabilize false vacuum
- Take T → ∞ limit
- Deform back and compute imaginary part

Is the result the decay rate?



3
- **Direct approach** using Minkowski space causal propagators
- Does not rely on saddle-point approximation
- Does not rely on deforming potential
- QFT derivation is simple – no bold leap of faith
- Non-perturbative formula

$$\Gamma_R = 2\text{Im} \lim_{T\to\infty} \left(\frac{\int_{x(-\mathcal{T})=a}^{x(\mathcal{T})=a} \mathcal{D}x \, e^{-S_{\text{E}}[x]} \delta(\tau_b[x])}{\int_{x(-\mathcal{T})=a}^{x(\mathcal{T})=a} \mathcal{D}x \, e^{-S_{\text{E}}[x]}} \right)_{\mathcal{T}\to iT}$$

Summary of potential deformation method

1. Deform the potential so FV is true ground state
2. Take T → ∞
   - Picks out $E_{FV}(g)$
   - Fixes integration contour to be the steepest descent contour passing through the static FV saddle point
3. Deform back

$T << T_{NL}$ (no return flux)

$T >> T_{slosh}$ (only metastable FV decay)

OR

- Compute Z by integrating along the steepest descent contour passing through the static FV saddle point

OR

- Compute Γ by integrating along the steepest descent contour passing through the bounce, taking the imaginary part, and multiply by 1/2

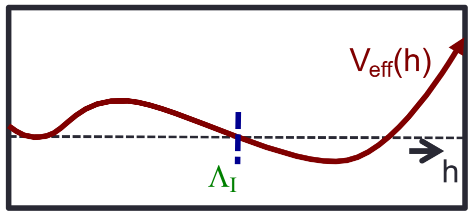
- Mathematically consistent procedure to get imaginary part out of an analytic real function Z
- Has the right ingredients associated with the necessary limits

Does this procedure give the decay rate?

Sensitivity to new physics

Old way:

when is $\Lambda_I = \Lambda_{NP}$?

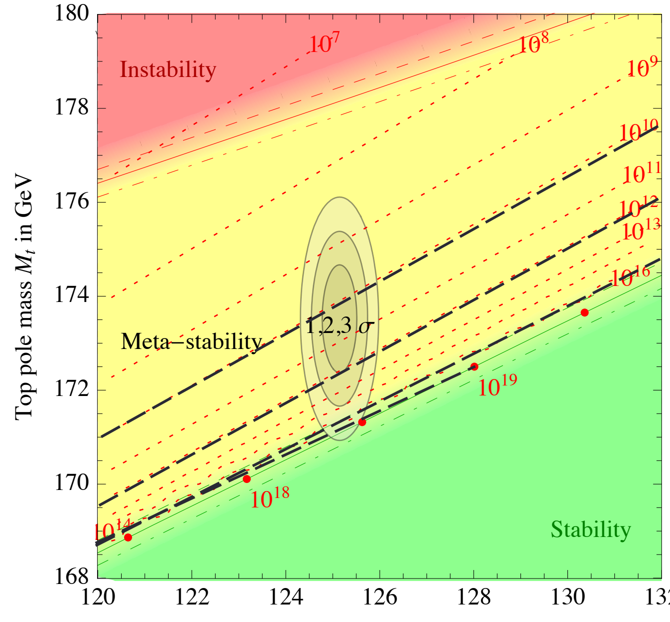


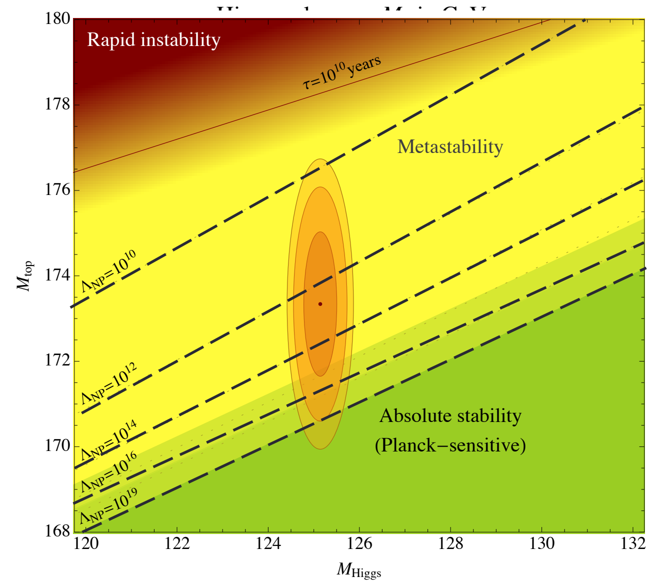
- gauge dependent, since Λ_I is gauge-dependent

New gauge-invariant way

- Add $\mathcal{O}_6 = \frac{1}{\Lambda_{NP}^2} |H|^6$ to the SM Lagrangian
- See how big Λ_{NP} must be so that $V_{min} = 0$

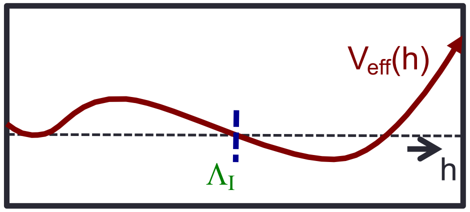
From Buttazzo et al (arXiv:1307.3536)





Planck-sensitivity

Does the tunneling rate depend on quantum gravity?

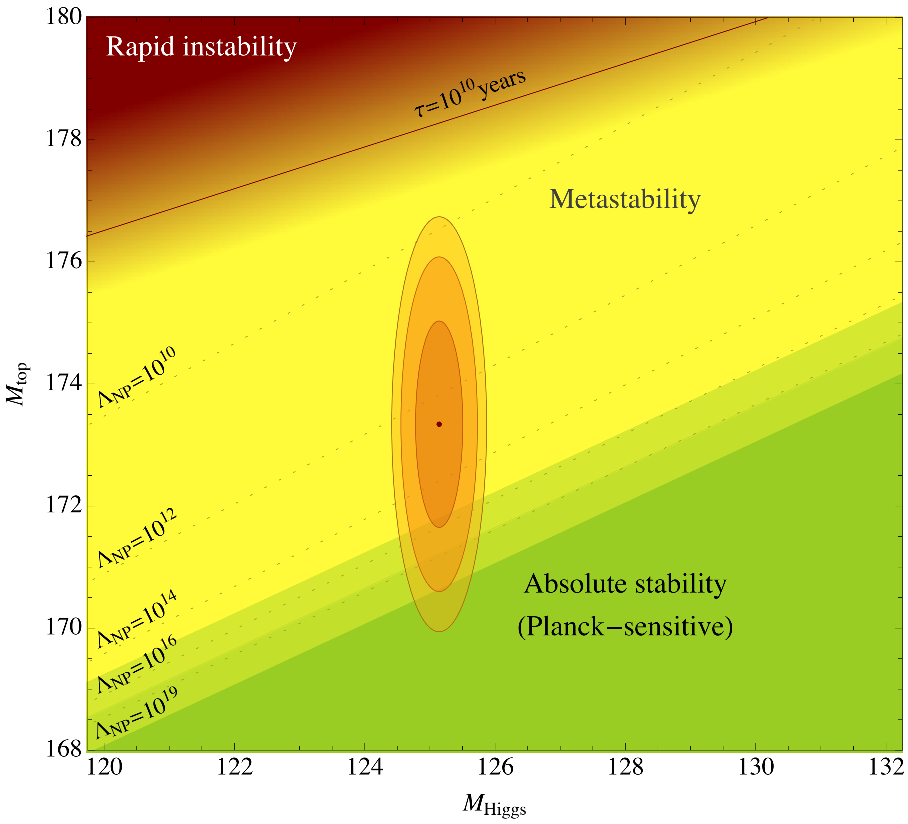


- Guidice, Strumia et al (arXiv:1307.3536):
  - Instability scale below Mpl, so **no**.

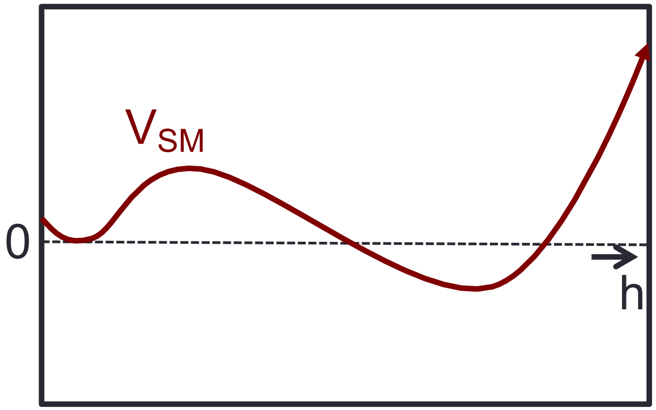
  $\beta_\lambda = 0$ at $\mu = 10^{17}$ GeV < M_{Pl}

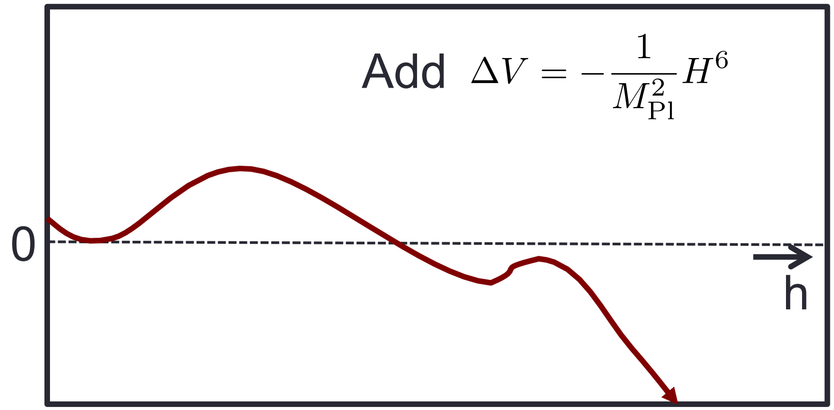
- Sher, Brandina et al (arXiv:1408.5302): field at center of bubble is greater than Mpl, so **yes**

$$\phi_B(r=0) = 10^{19}\mathrm{GeV} \sim M_{\mathrm{Pl}}$$



M_{Pl} corrections:

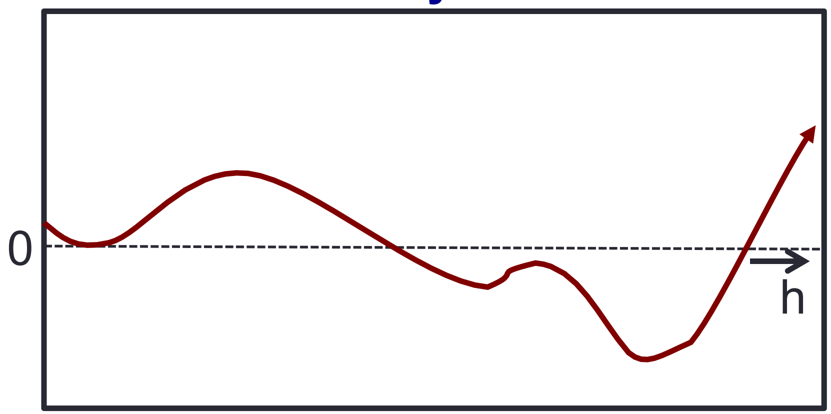




Standard Model potential

Liftetime = 10^{600} years

- **Lifetime = 0 sec**
- Arbitrarily small bubbles form and grow



Add $\Delta V = -\alpha \frac{1}{M_{\rm Pl}^2} H^6 + \beta \frac{1}{M_{\rm Pl}^2} H^8$

- **Lifetime can be anything**!

- Planck sensitivity not due to coincidence that $\beta_\lambda = 0$ at $\mu \sim M_{Pl}$
- Tunneling is **non-perturbative** and ***always*** UV sensitive.