AMPLITUDES, JETS AND ANTI-DESITTER SPACE

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Outline

- Why is perturbative QCD difficult?
- Infrared divergences, factorization and SCET
- Insights from anti-deSitter space
- Conclusions

Why is perturbative QCD difficult?

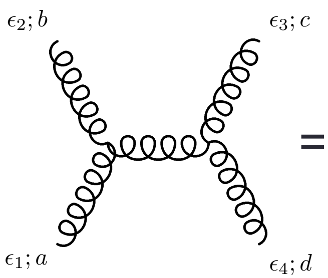
1. **Too many** Feynman diagrams
   - # diagrams grows factorially with # legs and # loops

2. Loop integrals are **difficult**
   - Numerical approach foiled by IR divergences

3. Phase space complicated
   - Fixed-order perturbative QCD of **limited practical use**

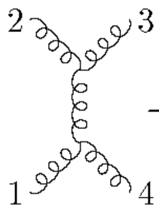
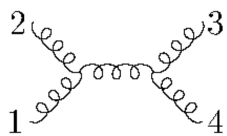
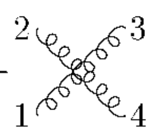
1. Too many Feynman diagrams

- Tree-level gg -> gg



$$\mathcal{M}_s(p_1 p_2 \to p_3 p_4) = \frac{g_s^2}{s} f^{abe} f^{cde}$$

$$= \times\{-4\,\epsilon_1\cdot\epsilon_3^\star\,\epsilon_2\cdot p_1\,p_3\cdot\epsilon_4^\star + 2\,\epsilon_1\cdot\epsilon_2\,\epsilon_3^\star\cdot p_1\,\epsilon_4^\star\cdot p_3 - 2\,\epsilon_1\cdot p_4\,\epsilon_2\cdot p_1\,\epsilon_3^\star\cdot\epsilon_4^\star + \epsilon_1\cdot\epsilon_2\,p_4\cdot p_1\,\epsilon_3^\star\cdot\epsilon_4^\star$$
$$+4\,\epsilon_1\cdot\epsilon_4^\star\,\epsilon_2\cdot p_1\,\epsilon_3^\star\cdot p_4 - 2\epsilon_1\cdot\epsilon_2\,\epsilon_3^\star\cdot p_4\,\epsilon_4^\star\cdot p_1 - 2\,\epsilon_1\cdot p_2\,\epsilon_2\cdot p_3\,\epsilon_3^\star\cdot\epsilon_4^\star + \epsilon_1\cdot\epsilon_2\,\epsilon_3^\star\cdot\epsilon_4^\star\,p_2\cdot p_3$$
$$+4\,\epsilon_1\cdot p_2\,\epsilon_2\cdot\epsilon_3^\star\,\epsilon_4^\star\cdot p_3 - 2\,\epsilon_1\cdot\epsilon_2\,\epsilon_2\cdot p_2\,\epsilon_4^\star\cdot p_3 + 2\,\epsilon_1\cdot p_2\,\epsilon_2\cdot p_4\,\epsilon_3^\star\cdot\epsilon_4^\star - \epsilon_1\cdot\epsilon_2\,\epsilon_3^\star\cdot\epsilon_4^\star\,p_4\cdot p_2$$
$$-4\,\epsilon_1\cdot p_2\,\epsilon_2\cdot\epsilon_4^\star\,\epsilon_3^\star\cdot p_4 + 2\,\epsilon_1\cdot\epsilon_2\,\epsilon_3^\star\cdot p_4\,\epsilon_4^\star\cdot p_2 + 2\,\epsilon_1\cdot p_3\,\epsilon_2\cdot p_1\,\epsilon_3^\star\cdot\epsilon_4^\star - \epsilon_1\cdot\epsilon_2\,\epsilon_3^\star\cdot\epsilon_4^\star\,p_1\cdot p_3\}$$

σ = |  +  +  |^2

Simple!

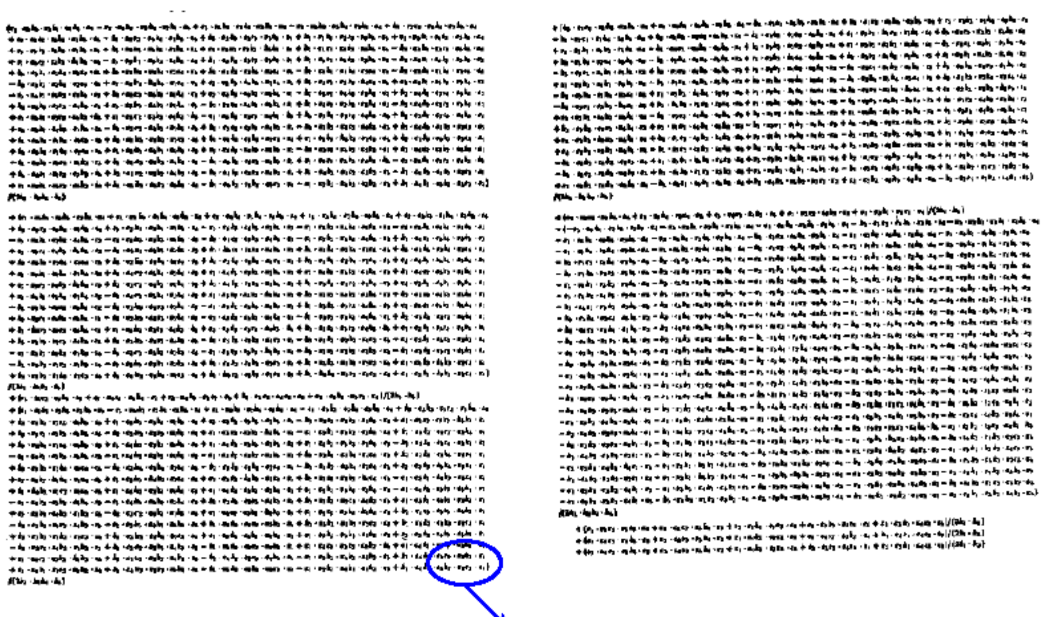
$$= g_s^4 \frac{9}{2}\left(3 - \frac{tu}{s^2} - \frac{su}{t^2} - \frac{st}{u^2}\right)$$

1. Too many Feynman diagrams

- Tree-level gg -> ggg

Result of a brute force calculation:

from Z. Bern

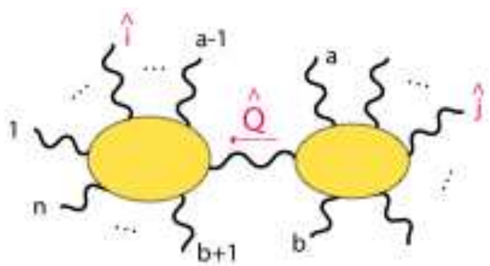


$k_1 \cdot k_4\, \varepsilon_2 \cdot k_1\, \varepsilon_1 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5$

1. Too many Feynman diagrams

Much progress in recent years

BCFW recursion relations



Parke-Taylor formula
(tree-level gluon scattering)

$$\widetilde{\mathcal{M}}(1^+2^+\cdots j^-\cdots k^-\cdots n^+) = \frac{\langle jk\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\cdots\langle n1\rangle}$$

How general is the simplicity?

- Generalized unitarity methods
- Twistor space
- Grassmannians
- Integrebality
- …

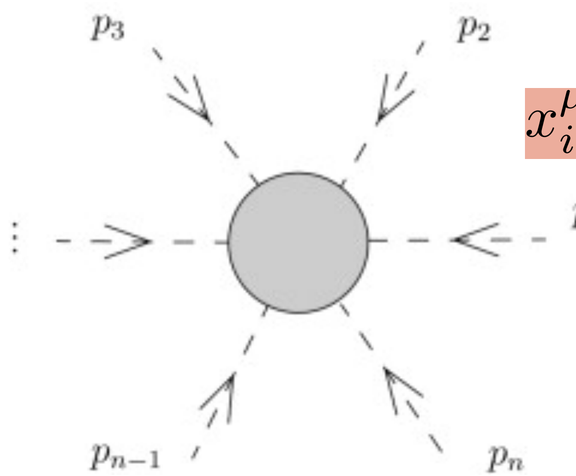
- Tree-level gluon scattering in QCD equal to tree-level gluon scattering in **N=4 SYM**

remains simple at all orders

Is simplicity special to N=4 SYM?

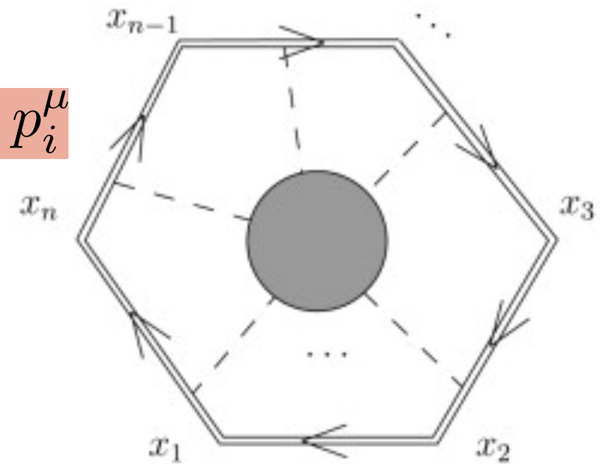
Dual conformal invariance

Symmetry of planar (large N) SYM theory



$$x_i^\mu = p_{i+1}^\mu - p_i^\mu$$

$\Longleftrightarrow$



Amplitudes invariant under $x_i^\mu \to \dfrac{x_i^\mu}{x_i^2}$ (inversion) and other conformal syms.

Conformal invariance of x_i called **dual conformal invariance**

Symmetry of S-matrix, not Lagrangian

Lagrangian description inadequate

Are there other hidden symmetries in gauge theories??

2. Loop integrals difficult

One loop : tensor reduction (Passarino-Veltman, 1979)

All loops = sum of master integrals

$$\text{Loop} = \sum_i (a_i A + b_i B + c_i C + d_i D) + \mathcal{R}$$

A, B, C, D are basis integrals: $B(p^2, m_1^2, m_2^2) = \int \frac{d^d k}{[k^2 - m_1^2 + i\epsilon][(k+p)^2 - m_2^2 + i\epsilon]}$

Problem 1: what are a_i, b_i, c_i, d_i?

- Unitarity methods
- Essentially solved

Problem 2: how to evaluate A,B,C,D?

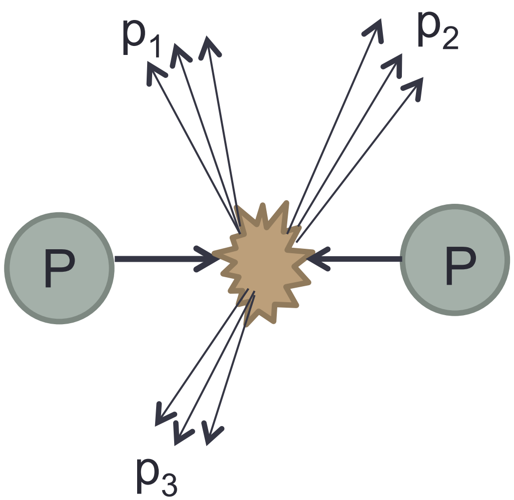
- For m=0 infrared divergences
- Analytical results known (2007)

Two loop and higher

- Basis unknown
- Unitary methods challenging
- Infrared divergences complicated

3. Phase space complicated

What is the differential 3-jet rate? $\dfrac{d\sigma}{dp_1 dp_2 dp_3}$



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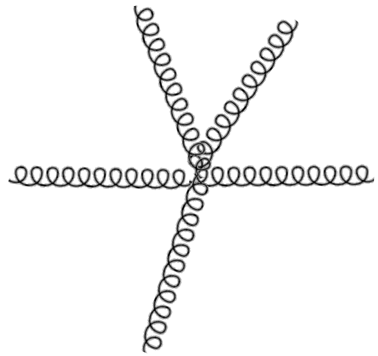


$$\sim \alpha_s \left(\overbrace{\cdots + \frac{(p_1+p_3)^2}{(p_2+p_1)^2} + \cdots}^{\text{Can become very large even if } \alpha_s \text{ is small}} \right)$$

$$(p_2+p_1)^2 \sim 0$$

Infrared divergent region

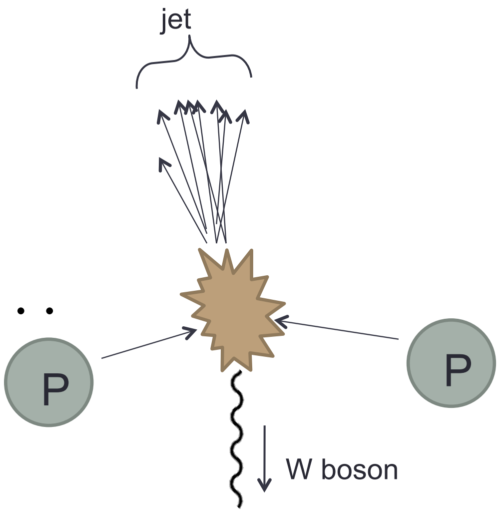
- Fixed order calculations only valid for **inclusive quantities**
  - Total rate for 3 well-separated jets
- **Exclusive distributions** (with hadrons) are needed for **experimental searches**
- Tails of QCD distributions often need for new physics searches
- **No natural** choice for factorization and renormalization **scales**

$\boxed{\text{Resummation critical}}$

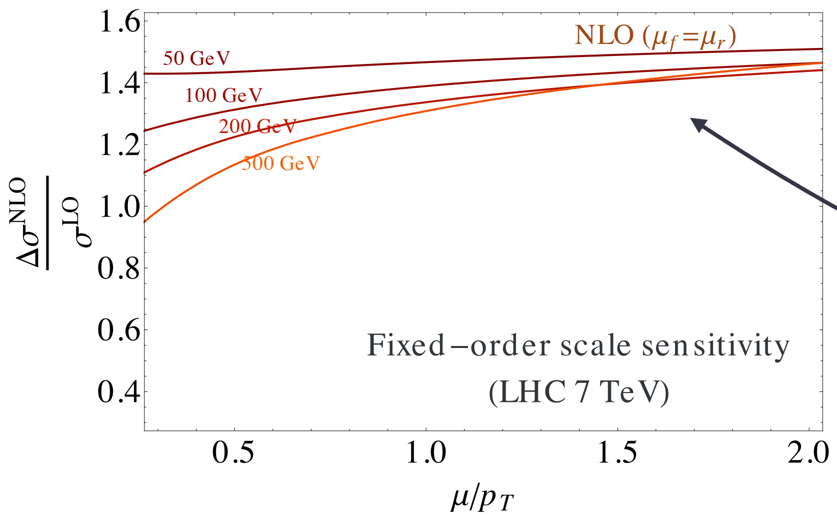
Scale choices derail pQCD

$$\frac{d\sigma}{dp_T} = \sum_{ij} \int f_i(x_1, \mu_f) f_j(x_2, \mu_f) \hat{\sigma}_{ij}(p_T, \mu_R)$$

$$\hat{\sigma} = \alpha_s(\mu_R) f_1(p_T, \mu_R) + \alpha_s^2(\mu_R) f_2(p_T, \mu_R) + \cdots$$



- Formally independent of scales when $\mu_f = \mu_R = \mu$
- Scale dependence induced by **truncation to fixed order**
- Scales chosen "intuitively" and varied by factors of 2



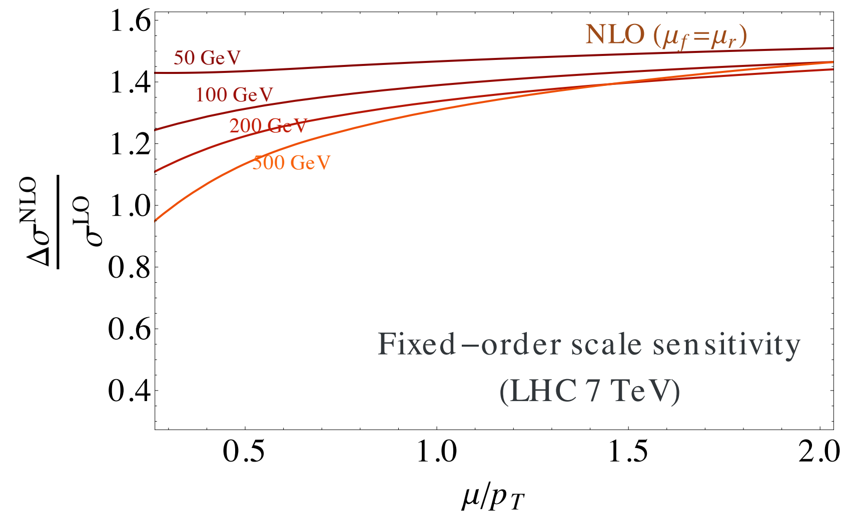
Monotonic dependence on scales

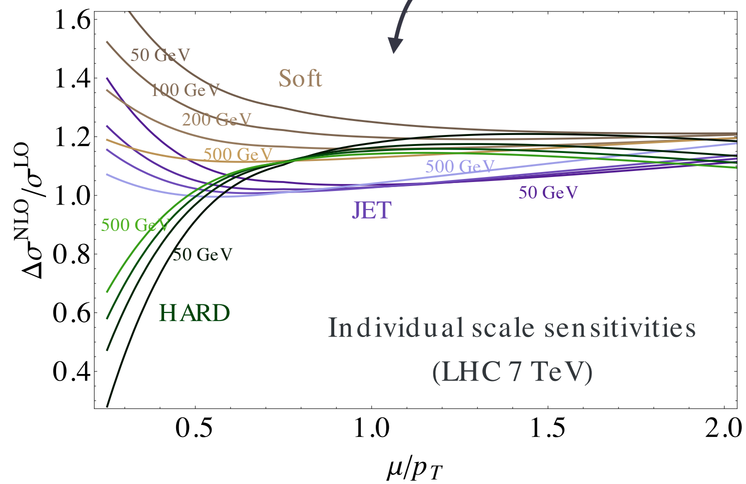
More scales are present

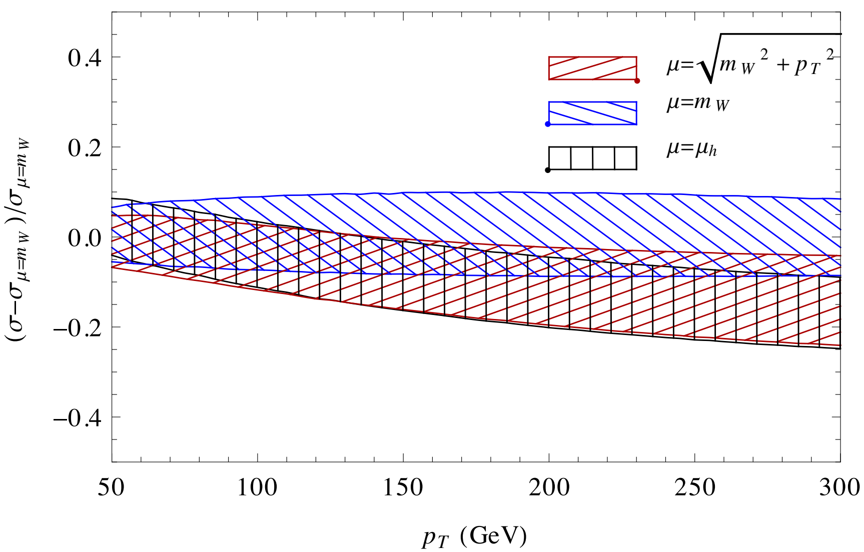
Becher, Lorentzen and **MDS**, Phys.Rev. D 86 (2012)

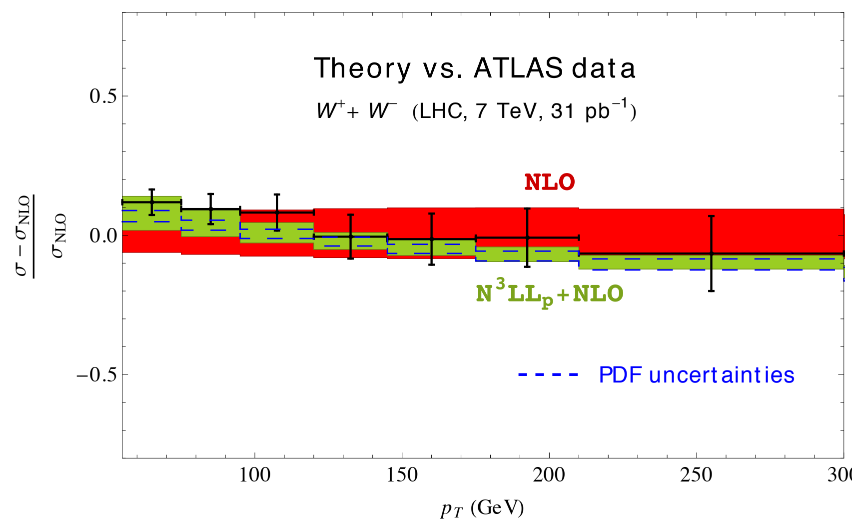
$$\sigma = f_1 \otimes f_2 \otimes H \otimes J \otimes S$$

Resummed calculation picks scales automatically

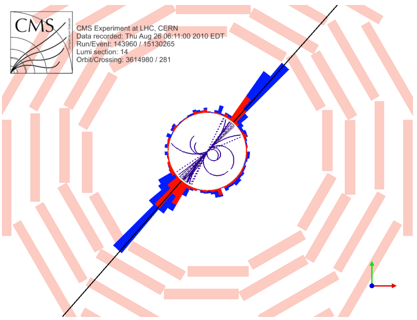








Resummation can be essential

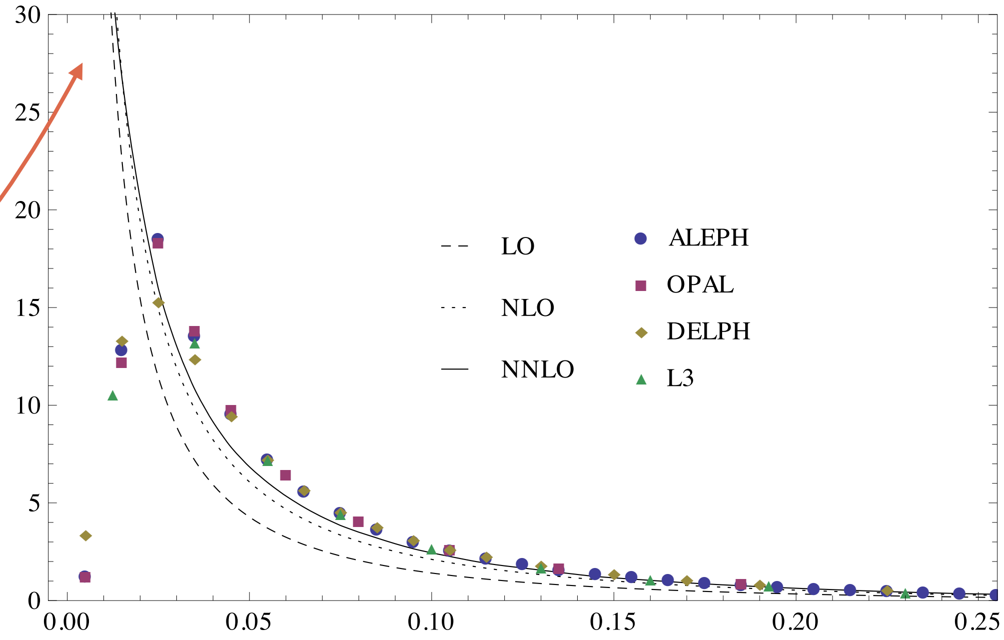


Suppose we want to know the distribution of **jet masses**.

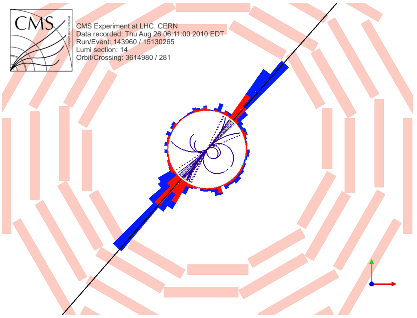
In QCD, for very jet-like jets

$$m^2 \frac{d\sigma}{dm^2} \approx 1 - \alpha_s \Gamma \ln^2 \frac{m^2}{E^2} + \frac{1}{2}\left(\alpha_s \Gamma \ln^2 \frac{m^2}{E^2}\right)^2 + \cdots$$

- **Blows up** as $m/E \to 0$
- QCD perturbation theory **breaks down**



Resummation can be essential



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Sudakov double logs

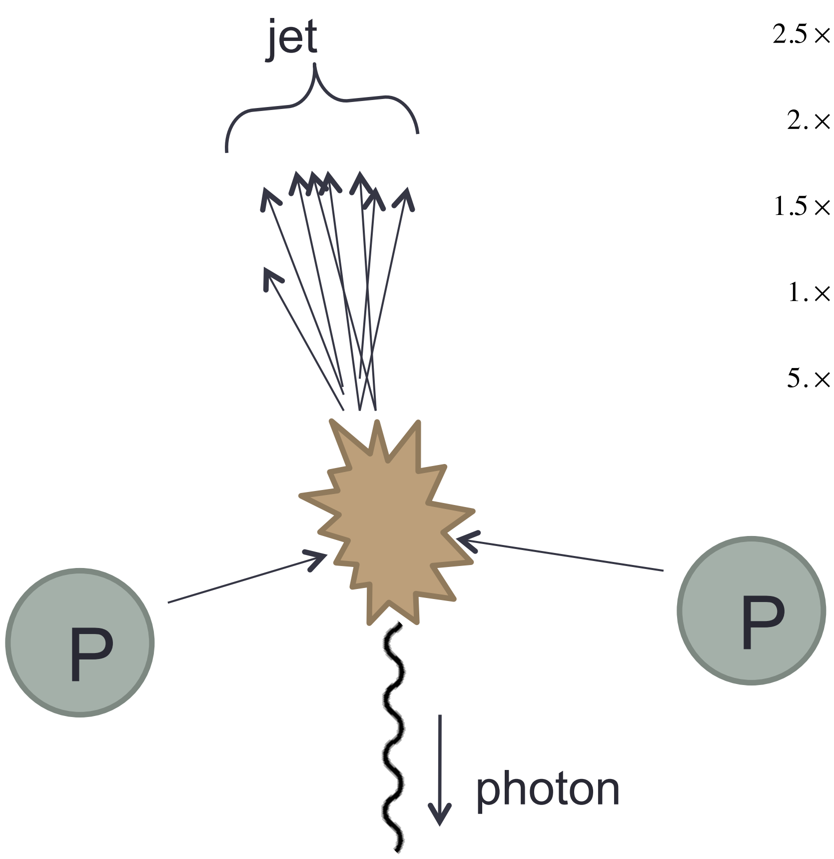
We can **sum the series**:

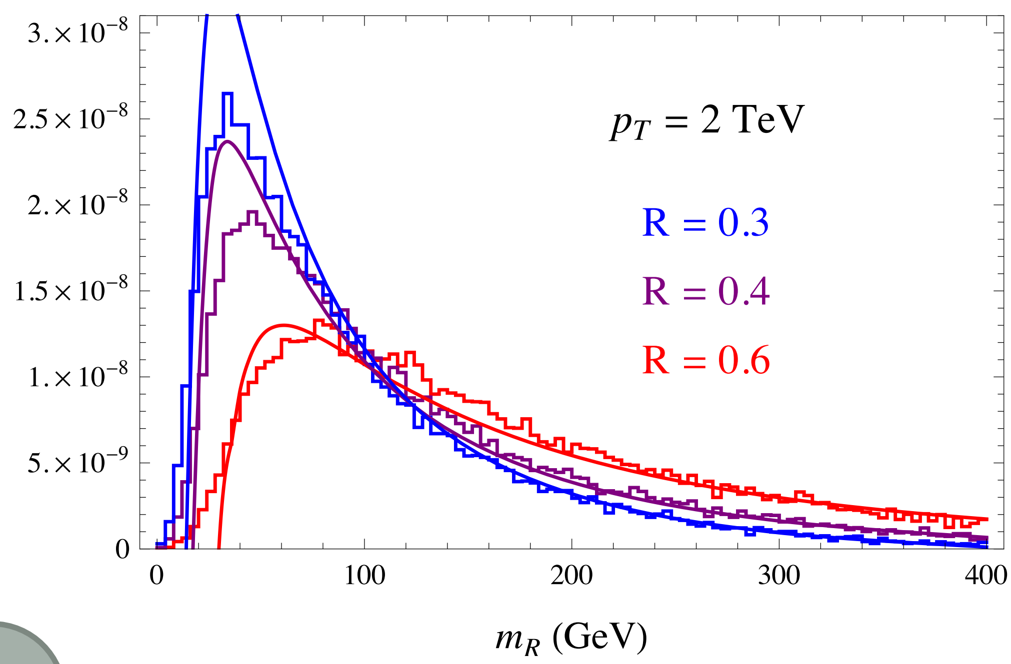
$$m^2 \frac{d\sigma}{dm^2} \approx e^{-\alpha_s \Gamma \ln^2 \frac{m^2}{E^2}}$$

- Now **convergent** as $m/E \to 0$

γ + jet at the LHC

Chien, Kelley, MDS and Zhu, **Phys. Rev. D** (2013, to appear)

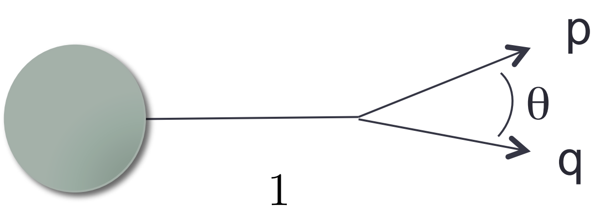




INFRARED DIVERGNCES AND WILSON LINES

Infrared divergences

- UV divergences known exactly for QCD (renormalizability)
- IR divergences known only up to 2-loops
  - Conjectures for 3-loops and up
- Structure of IR divergences needed for subtractions in numerical loop integrals
- Infrared singular regions dominate cross sections



$$\frac{1}{(p+q)^2}$$

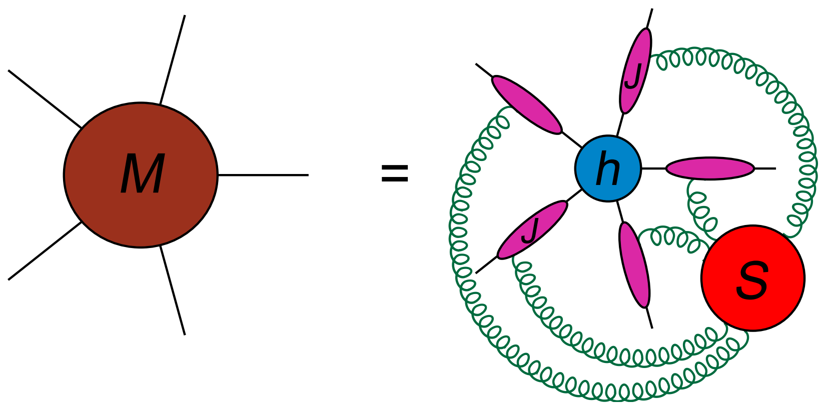
$$(p+q)^2 = E_q E_p \sin^2 \frac{\theta}{2}$$

soft singularity

collinear singularity

Soft-collinear singularity ⇨ Sudakov double logs

QCD factorizes in the infrared



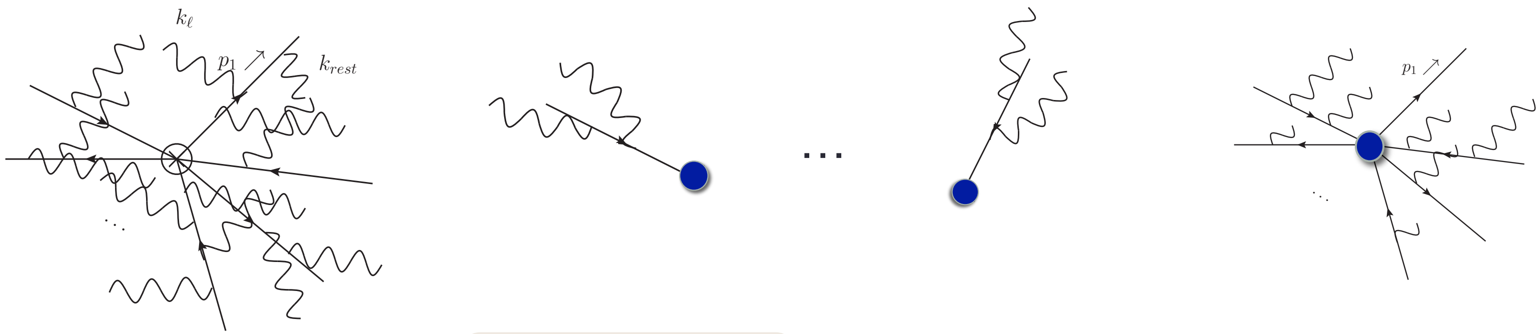
$$d\sigma = H \times J \otimes \cdots \otimes J \otimes S$$

Hard function

Jet functions

Soft function

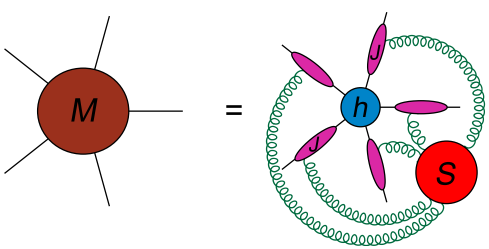
Precise statement of factorization



$$\langle 0|\bar{\psi}\cdots\psi|X\rangle \sim \langle 0|\bar{\psi}W_1|X_1\rangle\cdots\langle 0|W_n^\dagger\psi|X_n\rangle\langle 0|Y_1\cdots Y_n^\dagger|X_s\rangle$$

Wilson lines

$$\begin{cases} W_j = \mathcal{P}\left\{\exp\left(ig_s\int_{-\infty}^{0} ds t_j\cdot A(st_j^\mu)\right)\right\} \\ Y_j = \mathcal{P}\left\{\exp\left(ig_s\int_{-\infty}^{0} ds\, n_j\cdot A(sn_j^\mu)\right)\right\} \end{cases}$$



Describe radiation from moving charges

$$J = \sum_{X_c} |\langle 0|\bar{\psi}W|X_c\rangle|^2$$

$$S = \sum_{X_s} |\langle 0|Y_1\cdots Y_n|X_s\rangle|^2$$

Soft-collinear effective theory (SCET)

$$\langle 0|\bar\psi \cdots \psi|X\rangle \sim \langle 0|\bar\psi W_1|X_1\rangle \cdots \langle 0|W_n^\dagger \psi|X_n\rangle \langle 0|Y_1 \cdots Y_n^\dagger|X_s\rangle$$

or equivalently

$$\langle 0|\bar\psi \cdots \psi|X\rangle_{\mathcal{L}_{\rm QCD}} \sim \langle 0|\bar\psi W_1 Y_1 \cdots Y_n W_n \psi|X_1 \cdots X_n; X_s\rangle_{\mathcal{L}_{\rm SCET}}$$

$$\mathcal{L}_{\rm SCET} = \mathcal{L}_{\rm soft} + \sum_j \mathcal{L}_j$$

Identical copies of $\mathcal{L}_{\rm QCD}$

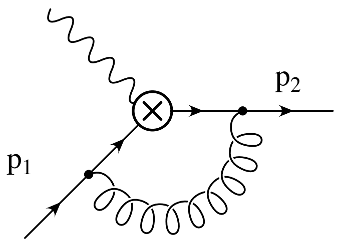
- Derivation using on-shell spinor-helicity methods

Feige and MDS, in preparation

- **Equivalent to** conventional **SCET** at leading power,
  but **much simpler**

Why is SCET efficient?

Loops involving fields in one sector are scaleless



$$= g_s^2 \mu^{4-d} \int \frac{d^d k}{(2\pi)^4} \frac{n_1 \cdot n_2}{(n_1 \cdot k)(n_2 \cdot k)} \frac{1}{k^2}$$

$$= \frac{1}{\epsilon_{\mathrm{IR}}^2} - \frac{1}{\epsilon_{\mathrm{UV}}^2} = 0$$

Double poles **in the IR**
(IR physics matches QCD)

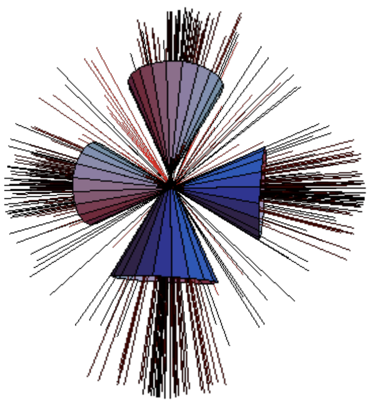
Double poles **in the UV**

- impossible in a local theory

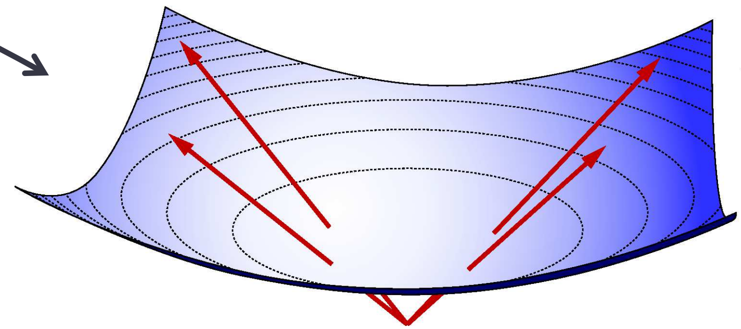
IR singularities of QCD extracted from UV divergences in SCET

- Anomalous dimensions give coefficients of IR poles
- Use **renormalization group** to resum Sudakov logarithms

Renormalization Group



1. Approximate jets with **Wilson lines**



2. Calculate **anomalous dimensions** of Wilson lines

$$\mu \frac{d}{d\mu} \mathcal{W} = \Gamma \, \mathcal{W}$$

$$\Gamma = \Gamma_{\text{cusp}}(\alpha_s) \log \mu + \gamma_{\text{reg}}$$

3. Resum Sudakov logs with **renormalization group**

$$m^2 \frac{d\sigma}{dm^2} \approx e^{-\alpha_s \Gamma \ln^2 \frac{m^2}{E^2}} = e^{-\alpha_s \int \Gamma \ln \frac{\mu}{E}}$$

ANOMALOUS DIMENSIONS OF WILSON LINES FROM ADS

Wilson line anomalous dimensions

Known results

1. Γ only depends on the angles between the lines
2. One loop result is: $\Gamma = -\frac{\alpha_s}{\pi} \sum_{i<j} \mathbf{T}_i \cdot \mathbf{T}_j ((\beta_{ij} - i\pi) \coth \beta_{ij} - 1)$
3. Abelian Exponentiation
4. In lightlike limit, anomalous dimension linearly on angles

Conjectures

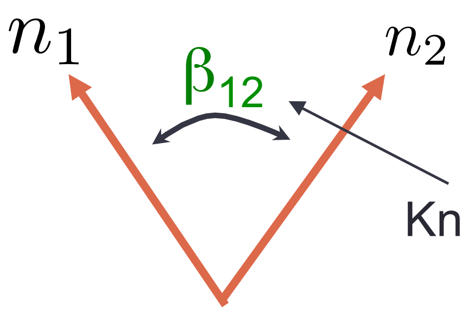
A. Pairwise Properties

B. Casimir Scaling

C. Absence of Conformal Cross-Ratios

Known results

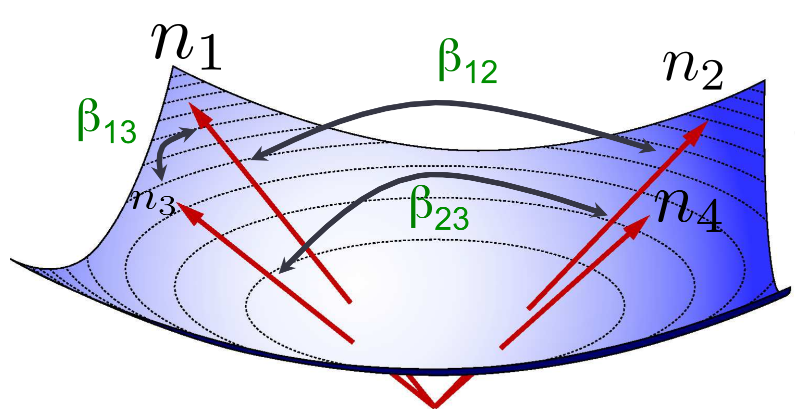
1. Γ only depends on the angles between the lines



Known as "cusp angle"

$$\cosh \beta_{ij} = \frac{n_i \cdot n_j}{|n_i||n_j|}$$

Multiple directions
Multiple cusp angles



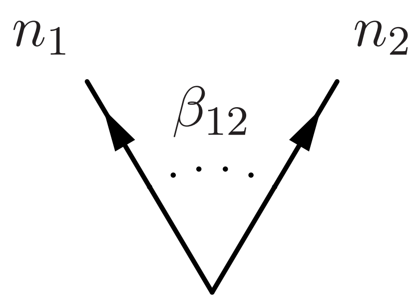
Lightlike limit: $n^2 \to 0 \implies \cosh \beta_{ij} = \frac{n_i \cdot n_j}{|n_i||n_j|} \to \infty$

$$\implies \beta_{ij} \to \infty$$

Known results

2. One loop result is:

$$\Gamma = -\frac{\alpha_s}{\pi} \sum_{i<j} \mathbf{T}_i \cdot \mathbf{T}_j ((\beta_{ij} - i\pi) \coth \beta_{ij} - 1)$$



Positive and **real** for outgoing/outgoing or incoming/incoming jets (e.g. $e^+e^- \to \text{jets}$)

Known results

3. Abelian Expnentiation

In QED, anomalous dimension is **exact** at 1-loop

Known results

4. At large angle, anomalous dimension depends linearly on cusp angles

$$\Gamma = -\sum_{i<j} \mathbf{\Gamma}^{ij}(\alpha_s)\beta_{ij}$$

True to **all orders** in perturbation theory

- For $n^2=0$, anomalous dimension is linearly **divergent**

$$\cosh\beta_{ij} = \frac{n_i \cdot n_j}{|n_i||n_j|} \to \infty$$

- In dimension regularization, this results in a linear dependence on log μ

$$\Gamma = -\sum_{i<j} \mathbf{\Gamma}^{ij}(\alpha_s)\log\left(\frac{n_i \cdot n_j}{\mu^2}\right)$$

Sudakov double logs

$$m^2\frac{d\sigma}{dm^2} \approx e^{-\alpha_s \Gamma \ln^2 \frac{m^2}{E^2}}$$

Wilson line anomalous dimensions

Known results

1. Γ only depends on the angles between the lines
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3. Abelian Expnentiation
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Conjectures

A. Pairwise Properties

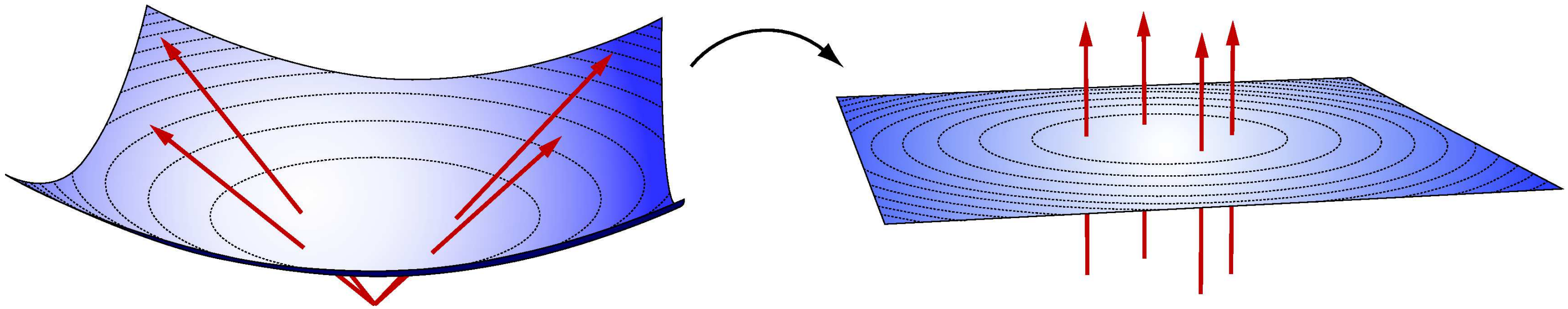
B. Casimir Scaling

C. Absence of Conformal Cross-Ratios

Is there a simpler way to understand these results?

Yes!

Change coordinates



Mapping to AdS

Dilatation operator

$$\mathcal{D} = x^\mu \partial_\mu = t\partial_t - r\partial_r$$

$$\mathcal{D}\mathcal{W} = \underbrace{(\Gamma + d)}_{\text{dimension}}\mathcal{W}$$

Anomalous dimension

New coordinates

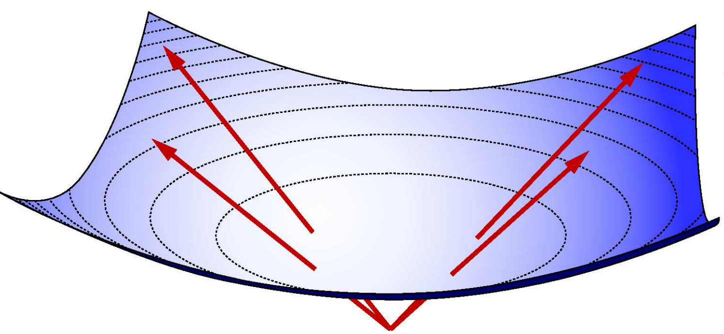
$$t = e^\tau \cosh\beta$$

$$r = e^\tau \sinh\beta$$

Dilatation maps to time translation

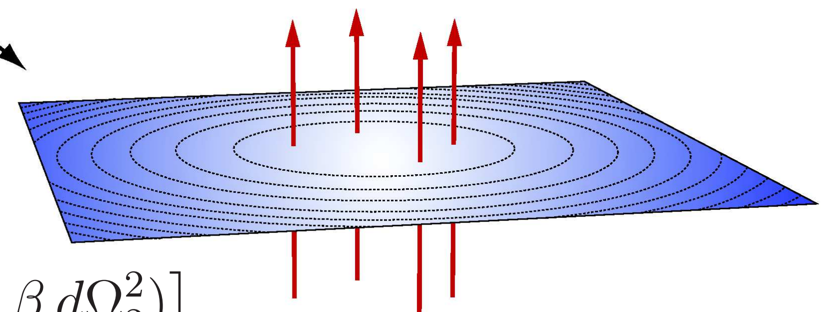
$$\frac{\partial}{\partial_\tau} = \frac{\partial t}{\partial \tau}\frac{\partial}{\partial t} - \frac{\partial r}{\partial \tau}\frac{\partial}{\partial r}$$

$$= t\partial_t - r\partial_r = \mathcal{D}$$



dimension **energy**

$$\mathcal{D}^{\mathbb{R}^{1,3}} = x^\mu \partial_\mu = \partial_\tau = i\mathcal{H}^{\mathbb{R}\times \mathrm{AdS}}$$



Euclidean AdS$_3$

$$ds^2_{\mathbb{R}^{1,3}} = dt^2 - dr^2 - r^2 d\Omega_2^2$$

$$= e^{2\tau}\left[d\tau^2 - (d\beta^2 + \sinh^2\beta\, d\Omega_2^2)\right]$$

Energies in AdS

What can energy depend on?

dimension energy

$$\mathcal{D}^{\mathbb{R}^{1,3}} = x^\mu \partial_\mu = \partial_\tau = i\mathcal{H}^{\mathbb{R}\times \mathrm{AdS}}$$

Homogeneous space $\Rightarrow$ Energy can only depend on geodesic distance between charges

Minkowski space

AdS

Two charges, in rest frame of charge 1

$$(t, r, \theta, \phi) \qquad \begin{aligned} t &= e^\tau \cosh\beta \\ r &= e^\tau \sinh\beta \end{aligned} \qquad (\tau, \beta, \theta, \phi)$$

$$n_1^\mu = (1, 0, 0, 0) \longrightarrow (0, 0, 0, 0)$$

$$n_2^\mu = (\cosh\beta_{12}, \sinh\beta_{12}, 0, 0) \longrightarrow (0, \beta_{12}, 0, 0)$$

$$n_1^2 = n_2^2 = 1$$

$$\frac{n_1 \cdot n_2}{|n_1||n_2|} = \cosh\beta_{12}$$

Definition of cusp angle

$$ds^2_{\mathbb{R}\times \mathrm{AdS}} = d\tau^2 - (d\beta^2 + \sinh^2\beta \, d\Omega_2^2)$$

$$\Delta s = \beta_{12}$$

Geodesic distance between charges is exactly the **cusp angle**

Wilson line anomalous dimensions

Known results

✓ 1. Γ only depends on the angles between the lines

2. One loop result is: $\Gamma = -\frac{\alpha_s}{\pi}\sum_{i<j}\mathbf{T}_i\cdot\mathbf{T}_j((\beta_{ij}-i\pi)\coth\beta_{ij}-1)$

3. Abelian Expnentiation

4. In lightlike limit, anomalous dimension linearly on angles

Conjectures

A. Pairwise Properties

B. Casimir Scaling

C. Absence of Conformal Cross-Ratios

What is the energy?

At leading order, just classical electrodynamics.
We can just solve Laplace's equation with $J_\tau = \delta^3(x)$ and $\vec{J} = 0$

$$\frac{1}{\sinh^2\beta}\partial_\beta(\sinh^2\beta(\partial_\beta A_\tau)) = J_\tau$$

and add the solutions linearly. $\Rightarrow$ $E_{\text{pair}}(\beta_{12}) = \frac{q_1 q_2}{4\pi^2}\Big[(\pi + i\beta_{12})\coth\beta_{12} + C\Big]$

Boundary condition: no cusp (forward scattering), conserved current, Γ=0

AdS result $\boxed{E_{\text{tot}} = \frac{i\alpha_s}{\pi}\sum_{i<j}\boldsymbol{T}_i\cdot\boldsymbol{T}_j\Big[(\beta_{ij} - i\pi)\coth\beta_{ij} - 1\Big]}$

Agrees exactly with 1-loop result
(Korchemsky & Radyushkin)

$$\Gamma = -\frac{\alpha_s}{\pi}\sum_{i<j}\boldsymbol{T}_i\cdot\boldsymbol{T}_j\,((\beta_{ij} - i\pi)\coth\beta_{ij} - 1)$$

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Abelian exponentiation

3. In QED (without dynamical matter), anomalous dimension is 1-loop exact

This is trivial in AdS since there are no loops.

- Energy of classical currents is exact at leading-order in QED

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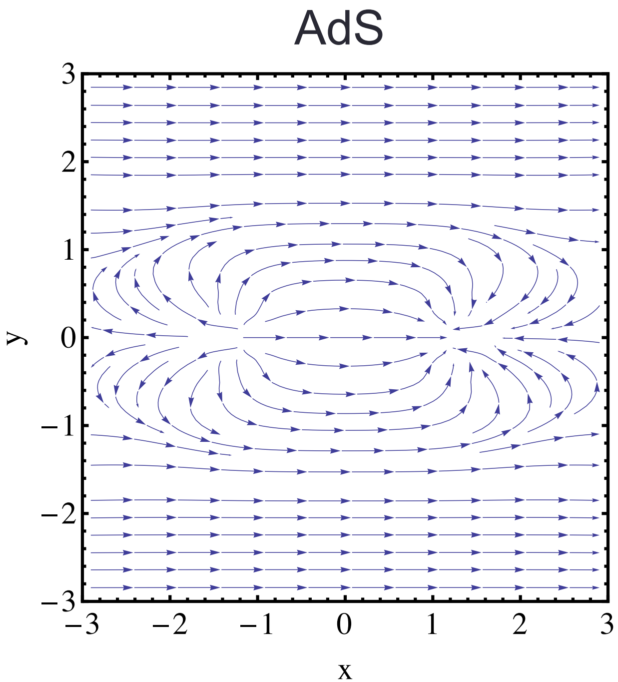
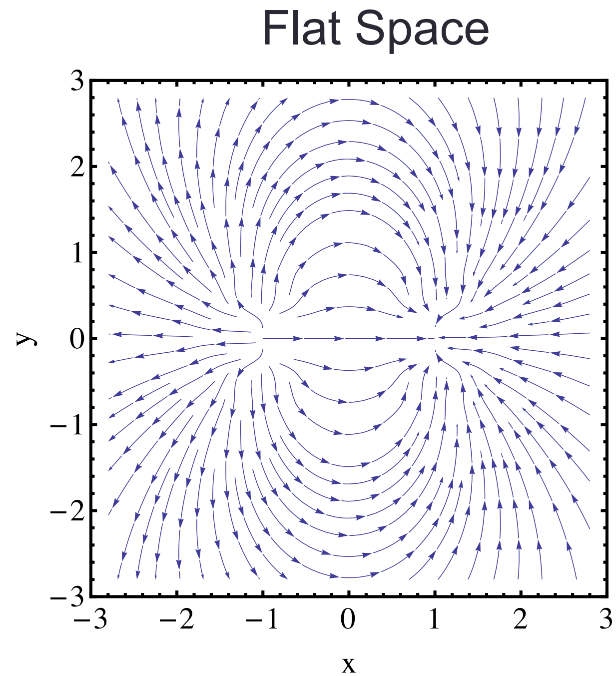
Linearity

Electric field lines



Linear at large separations



Vanishes at large separations

Energy is imaginary: **Decay**!

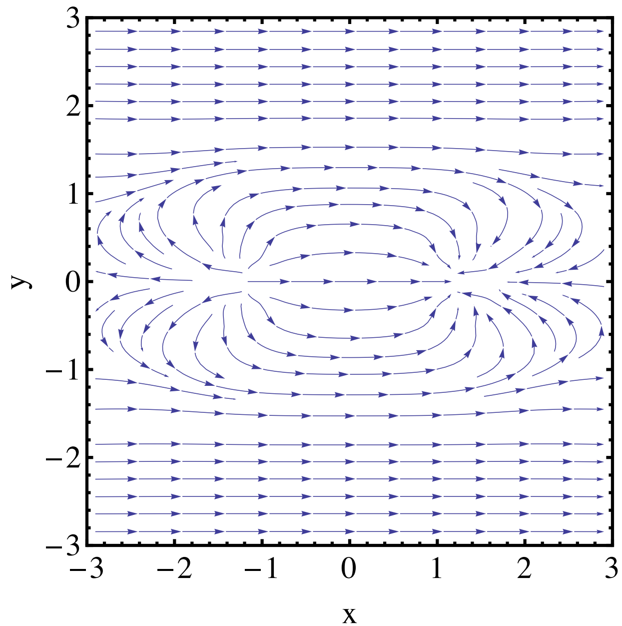
- Charges decay into radiation -- parton shower
- No free charges – confinement of parton within jet

Sudakov confinment

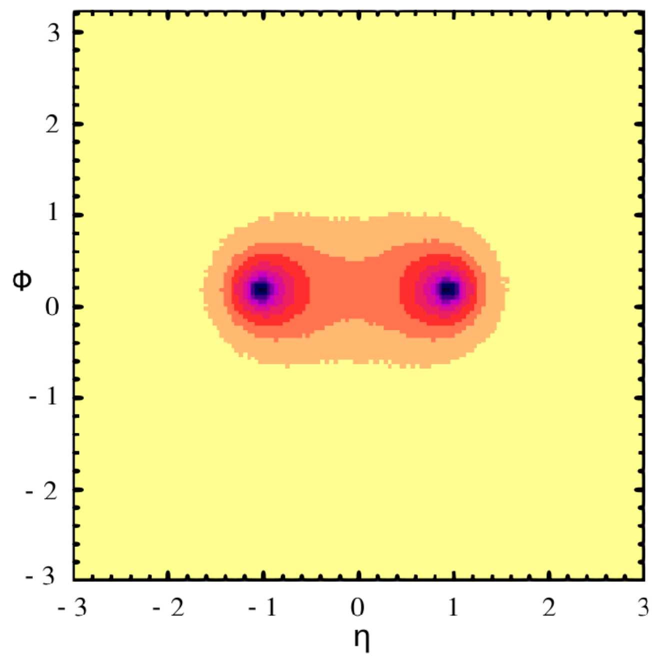
Linearity

Radiation pattern from **dipole**
Similar to distribution of **energy density**

AdS



Linear at large separations



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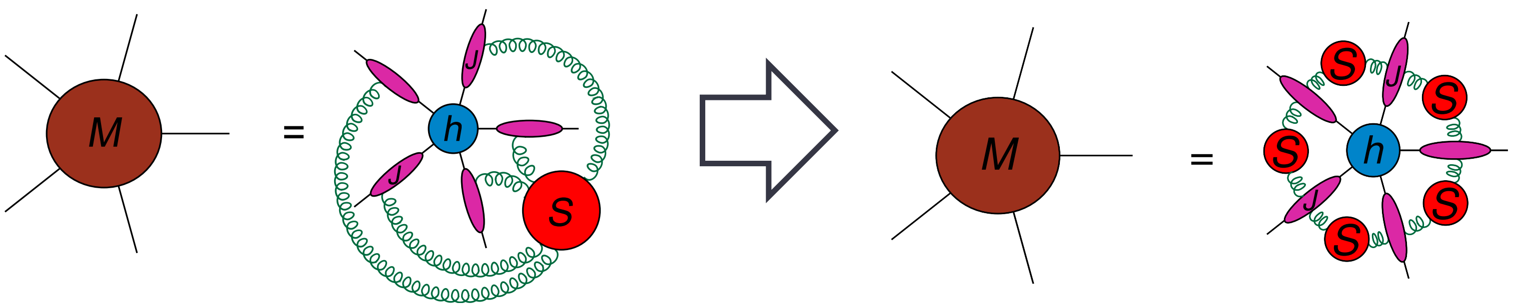
Conjectures

A. Pairwise Properties

$$\boldsymbol{\Gamma}^{ij}(\alpha_s) = \Gamma^{ij}(\alpha_s)\mathbf{T}_i \cdot \mathbf{T}_j$$

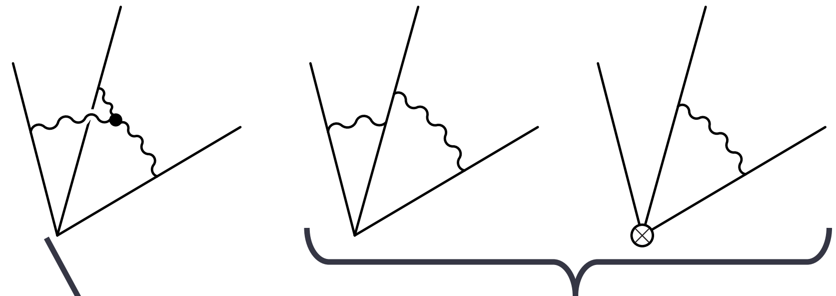


- Holds in planar limit
- Cusp anomalous dimension known exactly in planar N=4 SYM

Conjectures

A. Pairwise Properties

3 graphs at 2 loops

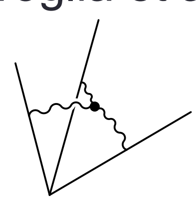


Sums of pairs

Should involve complicated color structures

$$\Gamma_{\mathrm{cusp}}^{2-\mathrm{loops}}(n_i) = \left(\frac{\alpha_s}{\pi}\right)^2 \left(\sum_{i<j} \boldsymbol{T}_i^a \boldsymbol{T}_j^a\, f(\gamma_{ij}) + \sum_{i<j<k} i f^{abc} \boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c\, F(\gamma_{ij}, \gamma_{jk}, \gamma_{ki})\right),$$

- **Vanishes** in the light-like case (Aybat, Dixon, Sterman)
- **Sum over pairs** for space-like case (Ferroglia et al)



$$= -\gamma_{ij}^2 - \frac{\pi^2}{4}$$

Conjectures

$$\mathbf{\Gamma}^{ij}(\alpha_s) = \Gamma^{ij}(\alpha_s)\mathbf{T}_i \cdot \mathbf{T}_j$$

B. Casimir Scaling

$$\mathbf{\Gamma}^{ij}(\alpha_s) = \Gamma^{\mathrm{cusp}}(\alpha_s)$$

- Holds at 3 loops, by direct calculation (Vogt, Moch, Vermaseren)
- General arguments prove at 3 loops and almost at 4 loops (Becher, Neubert, Gardi, Magnea, …)
- Violated in N=4 SYM at strong coupling (Amoni)
  - May still hold in perturbation theory

Conjectures

C. Absence of conformal cross ratios

$$\Gamma = -\sum_{i<j} \mathbf{\Gamma}^{ij}(\alpha_s)\beta_{ij} + \gamma_{\text{reg}}\left(\alpha_s, \cancel{\frac{(n_i \cdot n_j)(n_k \cdot n_l)}{(n_i \cdot n_k)(n_j \cdot n_l)}}\right)$$

- Can first appear with 4 wilson lines at 3 loops

- **General arguments** forbid almost all possible forms
  - Constrained by soft-collinear factorization (Becher, Neubert, Dixon, Gardi, Magnea)
  - Must vanish in collinear limits
  - Some possible forms found (Dixon et al)
  - Constraints from Regge limit (Gardi et al.)
  - Still an open question

- Would imply some **symmetries** that we are **missing**… very exciting!

Wilson line anomalous dimensions

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Conjectures

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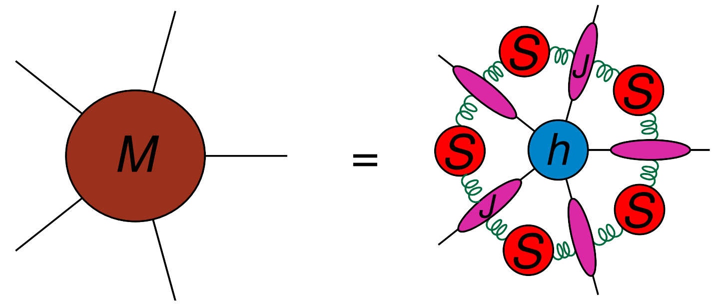
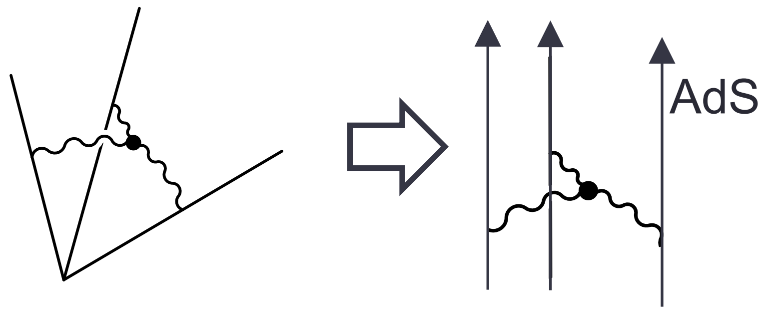
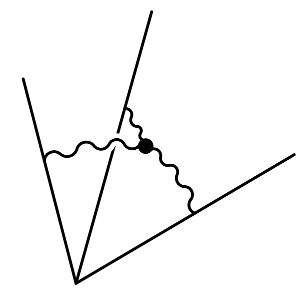
Pairwise properties





Currents in the time direction
only source scalar A_τ

- No scalar³ vertex in QCD $f^{abc} A^a_\tau A^b_\tau \partial_\tau A^c_\tau = 0$

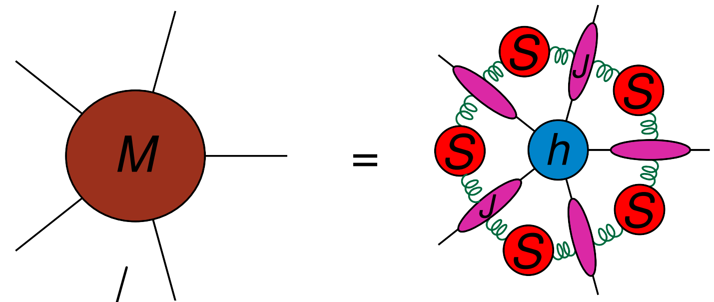


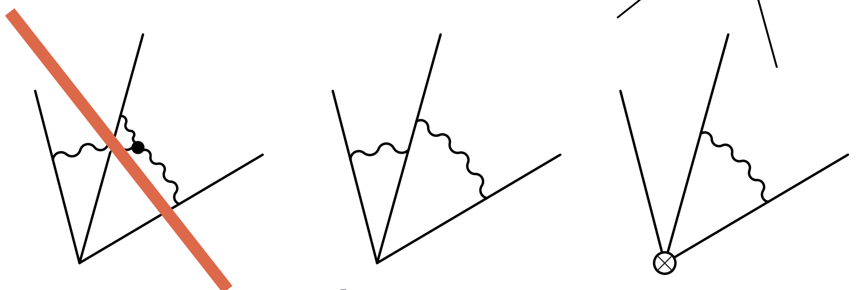
= 0

conformal gauge = Feynman gauge in AdS

- complicated non-local gauge in Minkowski space

Pairwise properties





Vanishes in conformal gauge

Already sums of pairs

For these graphs, conformal gauge gives the same result as Feynman gauge up to order ε

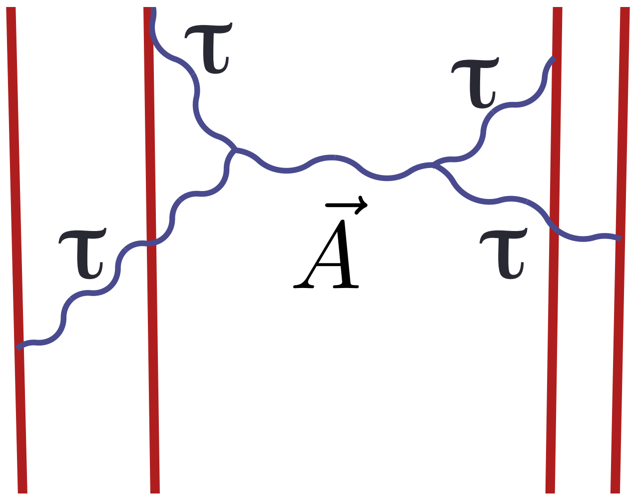
- Cross term from counterterm graph and O(ε) piece gives

$$\int_{-\infty}^{\infty} d\tau \log\left(\frac{\cosh\tau}{\cosh\tau+\cosh\gamma_{ij}}\right) = -\gamma_{ij}^2 - \frac{\pi^2}{4}\,.$$

Full result **agrees exactly** with Ferroglia et al.

- They needed integral reduction with Mellin-Barnes!

3-loops



- No reason it should be pairwise
- Pairwise structure accident at 2-loops

Known results

✓ 1. Γ only depends on the angles between the directions

✓- 2. At large angle, anomalous dimension depends linearly on cusp angles

✓ 3. One loop result is: $$\Gamma = -\frac{\alpha_s}{\pi}\sum_{i<j}\mathbf{T}_i\cdot\mathbf{T}_j((\beta_{ij}-i\pi)\coth\beta_{ij}-1)$$

✓ 4. Abelian Expnentiation

and conjectures

✓ A. Pairwise Properties

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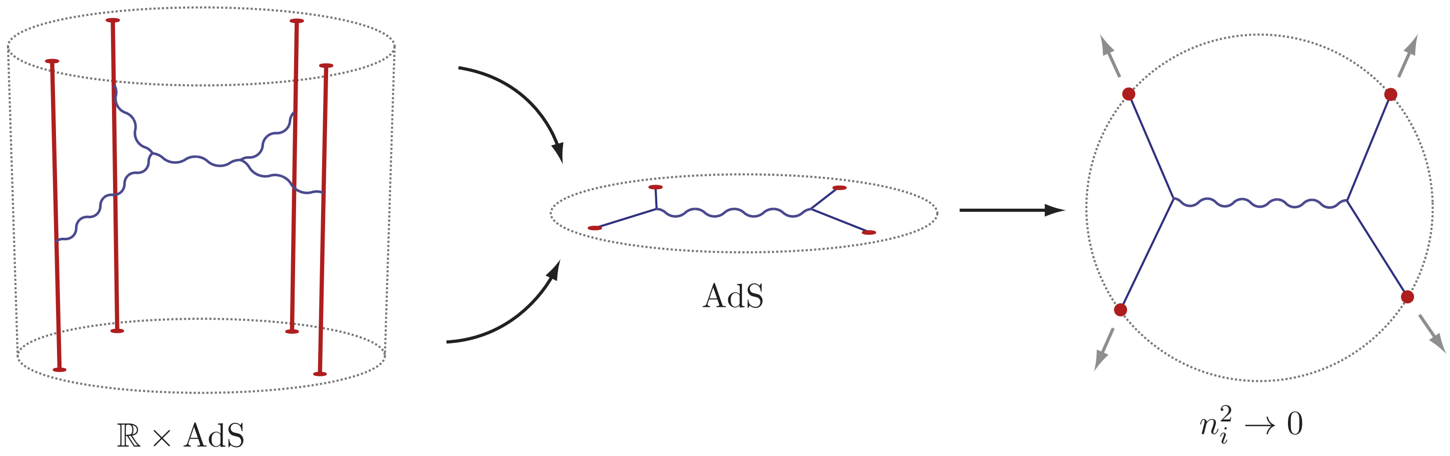
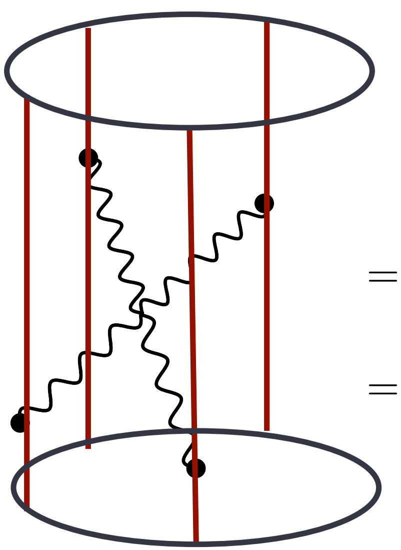
No conformal cross ratios?



Light-like limit
relates Wilson line loops to Witten diagrams

E.g.



$$= 8\Gamma(3-d/2)\frac{1}{n_{13}n_{24}}\overline{D}_{1111}(u,v)$$

$$= 8\Gamma(3-d/2)\frac{1}{n_{13}n_{24}}\frac{z\bar{z}}{z-\bar{z}}\left[2\mathrm{Li}_2(z)-2\mathrm{Li}_2(\bar{z})+\log(z\bar{z})\log\frac{1-z}{1-\bar{z}}\right]$$

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and conjectures

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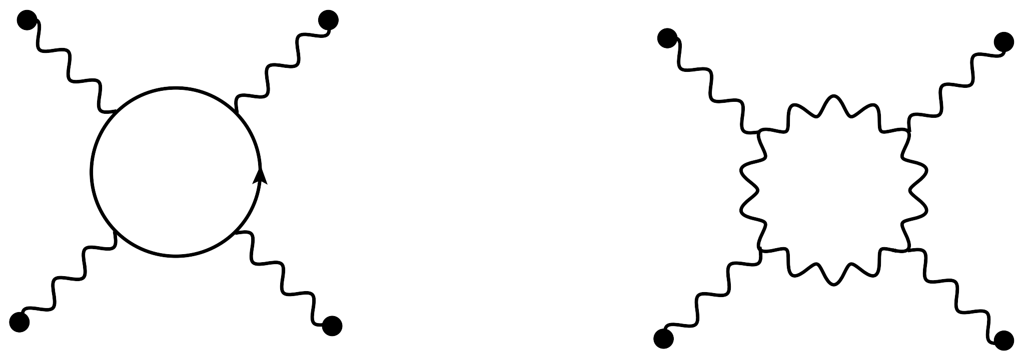
? C. Absence of Conformal Cross-Ratios

Casimir scaling

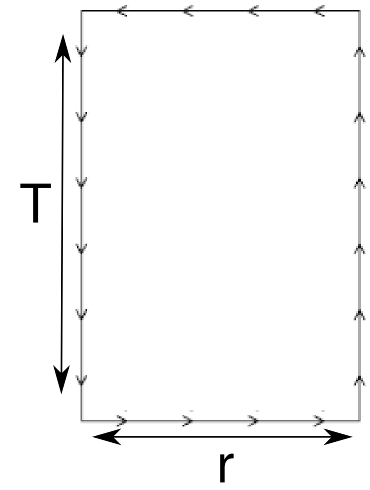
First violation could occur at 4-loops….



Energy interpretation

$$V(R) = \lim_{T\to\infty} \frac{1}{iT} \ln\langle W_{\rm loop}\rangle$$
$$= C_F \frac{\alpha_s}{r} + \mathcal{O}(\alpha_s^2)$$

$W_{\rm loop}$



- Casimir scaling violated at 3-loops (Sumino et al, 2010)
- Calculation done in Coulomb gauge.
- Equivalent calculation in **conformal gauge** would indicate Casimir-scaling violation for $\Gamma_{\rm cusp}$

Conclusions

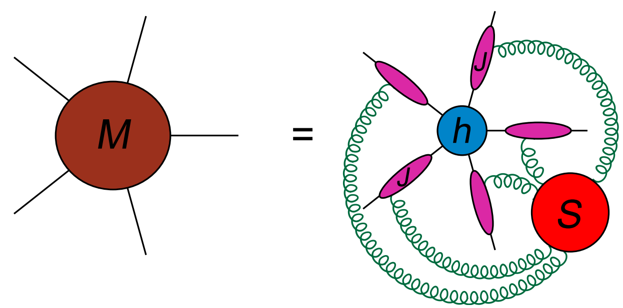
Perturbative QCD is no longer about computing Feynman diagrams!

Amplitudes are simpler than individual diagrams

On-shell methods providing practical results

Resummed results (all orders in α_s) critical for the LHC

Soft and collinear regions dominate cross sections



- Mapping to AdS helps understand infrared structure