Matthew Schwartz
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Lecture 12: Bose-Einstein Condensation

1 Introduction

Bose-Einstein condensation is a quantum phenomenon in Bose gases in which a large number of bosons simultaneously occupy the ground state of a system. Bose-Einstein condensates were predicted in 1925 by Bose and discovered experimentally 70 years later by Weimann, Cornell and Ketterle who shared the Nobel prize for their discovery in 2001. More precisely, these scientists constructed an experiment where the phase transition to Bose-Einstein condensation could be clearly seen and measured. Many quantum phenomena such as superconductivity, superfluids or lasers can also be understood as Bose-Einstein condensation.

Your first thought might be, of course a lot of bosons are in the ground state! After all, there is no quantum effect preventing them from being in the ground state (no Pauli exclusion), and naturally particles want to be in the state of lowest energy. This is a good thought; let's follow through. How many bosons do you expect in the ground state? Well, say they obey Maxwell-Boltzmann statistics, so that $n_i \sim e^{-\varepsilon_i / k_B T}$. This function is pretty flat for $\varepsilon_i \ll k_B T$, so we would expect that if there are say 100 states below $k_B T$ then each one should have roughly the same number of particles in it – nothing too special about the ground state. Thus, if you want a sizable fraction, say 1/2 the particles, to be in the ground state, you would have to get $k_B T$ down below the energy of the first excited state ε_1. This argument is correct for Maxwell-Boltzmann statistics, and we'll flesh it out more in a moment. The amazing thing is that with Bose-Einstein statistics the argument completely fails – you can find more than half of the particles in the ground state even for temperatures with $k_B T \gg \varepsilon_1$.

Bose-Einstein condensation is tricky to explain, so we'll approach it different ways. First, we'll try to understand what it is about Bose-Einstein statistics that allows condensation to happen through a simple system that we can solve in the canonical ensemble. Then we'll do the general case using the grand-canonical ensemble, first numerically, and then through analytic expansions.

In this lecture, it will be helpful to set the ground state energy to zero: $\varepsilon_0 = 0$. By setting it to zero, we mean that we list all energies as *relative* to the ground state. Shifting all the energies as well as the chemical potential in this way will have no effect on the physics and can be done without loss of generality.

2 Two-state system: canonical ensemble

Consider a system with N particles but only two possible energy states: $\varepsilon_0 = 0$ and $\varepsilon_1 = \varepsilon$. Because there are only two states, we can study this system in the canonical ensemble for both Maxwell-Boltzmann statistics and Bose-Einstein statistics. We'll see that even in this two-state system the ground state occupancy can be much larger than $\frac{N}{2}$ even when $k_B T \gg \varepsilon$ with Bose-Einstein statistics. With the large number of states present in any realistic system, the canonical ensemble will not be tractable and we will have to resort to the more abstract grand canonical ensemble.

Let's start with the simplest case of our 2-state system, $N = 1$. If there is only one particle, then for any statistics, the partition function is

$$Z_1 = \sum_k e^{-\beta E_k} = 1 + e^{-\beta \varepsilon} \tag{1}$$

The probability of finding the particle in the ground state is $P_{\text{ground}} = \frac{1}{Z_1} e^{-\beta \varepsilon_0} = \frac{1}{Z_1}$ and so the expected fraction of particles in the ground state is

$$\frac{\langle N_{\text{ground}} \rangle}{N} = 1 \cdot P_{\text{ground}} + 0 \cdot P_{\text{not-ground}} = \frac{1}{Z_1} = \frac{1}{1 + e^{-\beta \varepsilon}} \tag{2}$$

Again, this holds for any statistics, since there is only one particle.

Now say there are N particles. With Maxwell-Boltzmann statistics, the probability of finding any particle in the ground state is independent of the probability of finding any other particle anywhere. This implies that the N particle partition function is related to the 1 particle one by

$$Z_N^{\mathrm{MB}} = \frac{1}{N!}(Z_1)^N = \frac{1}{N!}(1+e^{-\beta\varepsilon})^N \tag{3}$$

Of the 2^N microstates, there is only one microstate with all the particles in the ground state, so

$$P_{\text{all ground}} = \frac{1}{Z_N^{\mathrm{MB}}}\left(\frac{1}{N!}e^{-\beta\varepsilon_0}\right) = \frac{1}{Z_N^{\mathrm{MB}}}\frac{1}{N!} \tag{4}$$

Note that we include the $\frac{1}{N!}$ in the probabilty for the same reason we do in Z_N^{MB}, to account for identifal particles. To confirm that this is correct, let us verify that the probabilities sum to one. There are N states with 1 particle in the excited state, $\binom{N}{2}$ states with 2 particles in the excited state and so on. So the sum of probabilties is

$$\sum_k P_k = \frac{1}{Z_N^{\mathrm{MB}}}\frac{1}{N!}\left[1 + Ne^{-\beta\varepsilon} + \binom{N}{2}e^{-2\beta\varepsilon} + \cdots + \binom{N}{N}e^{-N\beta\varepsilon}\right] = \frac{1}{N!Z_N^{\mathrm{MB}}}(1+e^{-\beta\varepsilon})^N = 1 \tag{5}$$

where Eqs. (1) and (3) were used in the last step. To compute the expected number in the ground state, we multiply each term in this sum by the ground state occupancy:

$$\langle N_{\text{ground}}^{\mathrm{MB}}\rangle = \frac{1}{N!Z_N^{\mathrm{MB}}}\left(N\cdot 1 + (N-1)\cdot Ne^{-\beta\varepsilon} + (N-2)\cdot\binom{N}{2}e^{-2\beta\varepsilon} + \cdots + 0\cdot\binom{N}{N}e^{-N\beta\varepsilon}\right) \tag{6}$$

$$= \frac{N}{e^{-\beta\varepsilon}+1} \tag{7}$$

You can check this sum in Mathematica. Note that at large T ($\beta\to 0$), $\langle N_{\text{ground}}^{\mathrm{MB}}\rangle$ goes to $\frac{N}{2}$: half the particles are in the ground state, half in the excited state.

With Bose-Einstein statistics there is only one state with m particles in the ground state and $N-m$ particles in the excited state. So there are only $N+1$ possible states all together and

$$Z_N^{\mathrm{BE}} = 1 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon} + \cdots + e^{-N\beta\varepsilon} = \frac{1-e^{-(N+1)\beta\varepsilon}}{1-e^{-\beta\varepsilon}} \tag{8}$$

Then the expected number in the ground state is

$$\langle N_{\text{ground}}^{\mathrm{BE}}\rangle = \frac{1}{Z_N^{BE}}[N\cdot 1 + (N-1)\cdot e^{-\beta\varepsilon} + (N-2)\cdot e^{-2\beta\varepsilon} + \cdots + 0\cdot e^{-N\beta\varepsilon}] \tag{9}$$

$$= \frac{1}{e^{-\beta\varepsilon}-1} + \frac{N+1}{1-e^{-(N+1)\beta\varepsilon}} \tag{10}$$

Let us look at Eqs. (7) and (10) numerically for $N=100$:

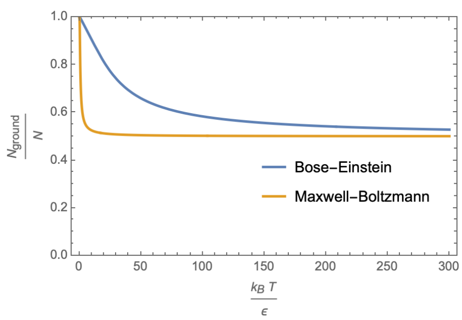


Figure 1. The fractional population of the ground state in a two state system with $N=100$.

This plot demonstrates Bose-Einstein condensation. With Maxwell-Boltzmann statistics, the temperature has to be very low to get the lowest state to have an appreciable filling fraction. At temperature $k_B T \gtrsim \varepsilon$ both the ground state and the first excited state are around equally populated so $\langle N_{\text{ground}}\rangle = \frac{N}{2}$. In contrast, with Bose-Einstein statistics, a significant fraction of the the particles are in the ground state even well above the temperature $k_B T = \varepsilon$. For example, with at $k_B T = 10\varepsilon$ we find 90% of the atoms are in the ground state for Bose-Einstein statistics, but only 52% for Maxwell-Boltzmann statistics. This demonstrates Bose-Einstein condensation.

Bose-Einstein condensation is a phase transition whereby the ground state become highly occupied. What is the critical temperature for this to occur? There is only two dimensionless numbers we can work with $\frac{k_B T}{\varepsilon}$ and N. We want N to be large, $N \gg 1$. Then we can expand in two limits $\frac{k_B T}{\varepsilon} \gg N$ (i.e. $\beta\varepsilon N \ll 1$) and $\frac{k_B T}{\varepsilon} \ll N$ (i.e. $\beta\varepsilon N \gg 1$). Expanding Eq. (10) in the first limit gives

$$\frac{\langle N_{\text{ground}}^{\text{BE}}\rangle}{N} = \frac{1}{2} + \frac{N\varepsilon}{12 k_B T} + \cdots \qquad \left(\frac{k_B T}{\varepsilon} \gg N\right) \tag{11}$$

This is the true high-temperature limit, where the classical behavior $\frac{\langle N_{\text{ground}}^{\text{BE}}\rangle}{N} \to \frac{\langle N_{\text{ground}}^{\text{MB}}\rangle}{N} \approx \frac{1}{2}$ is approached as $T \to \infty$. In the second limit $N \gg \frac{k_B T}{\varepsilon} \gg 1$, the temperature is large, but not too large, and the expansion gives a different result

$$\frac{\langle N_{\text{ground}}^{\text{BE}}\rangle}{N} = 1 - \frac{k_B T}{N\varepsilon} + \cdots \qquad \left(N \gg \frac{k_B T}{\varepsilon}\right) \tag{12}$$

This limit shows the growth of $\frac{\langle N_{\text{ground}}^{\text{BE}}\rangle}{N}$ toward 1 as $T \to 0$. The comparison of these approximations to Eq. (10) looks like

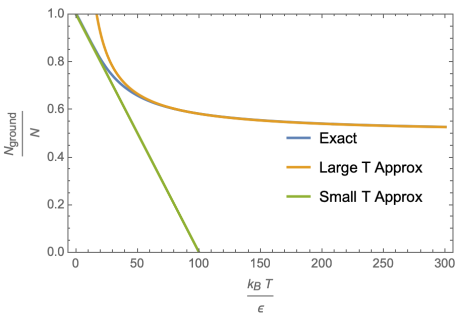


Figure 2. The approximations to the Bose-Einstein curve in Eqs. (11) and (12).

The crossover point is roughly where the approximations used for our expansions break down. The first term is the same order as the second term in Eq. (11) when $k_B T = \frac{N\varepsilon}{6}$. For Eq. (12) the crossover is at $k_B T = N\varepsilon$. Thus we find a critical temperature $T_c \sim \frac{N\varepsilon}{6k_B} \sim \frac{N\varepsilon}{k_B}$ for this two state model indicating the onset of Bose-Einstein condensation. (The crossover point in this 2-state example is not at a precise temperature, as you can see from the plot. T_C will become precise when we consider a realistic system with a large number of states in the next section.)

To emphasize how strange Bose-Einstein condensation is, remember that at $k_B T \gg \varepsilon$ we should be able to use $e^{-\beta\varepsilon} \approx 1$ independent of N. That is, the ground state and first excited state should have pretty similar thermodynamic properties and occupancy at high temperature. This is *not* what we are finding. Instead, for $N = 10$ million, with a temperature 1 million times the excited state energy, 90% of the atoms are in the ground state and only 10% are in the excited state.

Now, this 2-state model is not a realistic approximation to any physical system. It turns out to be very difficult to calculate $\langle N_{\text{ground}}^{\text{BE}}\rangle$ in the canonical ensemble for a realistic system that has an infinite number of states. The difficulty is that we have to count the number of ways of allocating N particles to the states, and then to perform the sum over occupancies of the ground state times Boltzmann factors. It turns out to be much easier to compute the general case using the grand-canonical ensemble with μ instead of N, as we will now see.

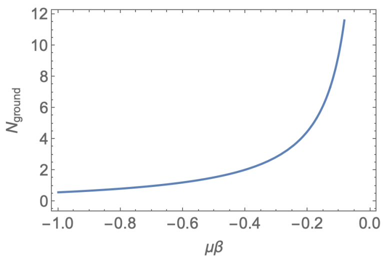
3 Grand canonical ensemble

With Bose-Einstein statistics, we determined that using the grand canonical ensemble the expected number of particles in a state i is

$$\langle N_i \rangle = \frac{1}{e^{\beta(\varepsilon_i - \mu)} - 1} \tag{13}$$

with ε_i the energy of the state i. So the expected number of particles in the ground state ($\varepsilon_i = 0$) is

$$\langle N_{\text{ground}} \rangle = \frac{1}{e^{-\beta\mu} - 1} = \tag{14}$$



Recall that for Bose gases, μ will always be lower then all the energies (i.e. $\mu < 0$ when $\varepsilon_0 = 0$). We can see this explicitly from the plot since there is a singularity at $\mu = 0$. This singularity, that $N_{\text{ground}} \to \infty$ as $\mu \to 0$ is not physical. Recall that in the grand canonical ensemble we do not fix N, so the same distribution has to allow for arbitrarily large N. If we know N then we have to trade μ for N by imposing the constraint $\sum_i \langle N_i \rangle = N$. This is not so easy, but we can do it.

To replace μ by N, we first invert Eq. (14) to solve for μ in terms of $\langle N_{\text{ground}} \rangle$:

$$e^{-\beta\mu} = \frac{1}{\langle N_{\text{ground}} \rangle} + 1 \tag{15}$$

Our strategy will then be to find the ground state occupancy by using the constraint the the total number of particles is N. That is, we will compute

$$N = \sum_{i=0}^{\infty} \langle N_i \rangle = \sum_{i=0}^{\infty} \frac{1}{e^{\beta\varepsilon_i} e^{-\beta\mu} - 1} = \sum_{i=0}^{\infty} \frac{1}{e^{\beta\varepsilon_i}\left(\frac{1}{\langle N_{\text{ground}} \rangle} + 1\right) - 1} \tag{16}$$

Once we work out the ε_i we can do the sum numerically or analytically and therefore find $\frac{\langle N_{\text{ground}} \rangle}{\text{N}}$.

Before beginning the calculation, let us quickly ask about the non-ground state occupancies. Since

$$\mu = -\frac{1}{\beta} \ln\left[\frac{1}{\langle N_{\text{ground}} \rangle} + 1\right] < -\frac{1}{\beta} \ln\left[\frac{1}{N} + 1\right] < 0 \tag{17}$$

μ is always negative and gets closest to zero when $\langle N_{\text{ground}} \rangle$ is largest. Conversely, $\langle N_{\text{ground}} \rangle$ is largest when μ is closest to zero as can be seen in Eq. (14).

Can excited states have a large number of particles in them? The explosion of particles in the ground state arose because if $\mu \to 0$ then $e^{-\beta\mu} \to 1$ and $\langle N_{\text{ground}} \rangle = \frac{1}{e^{-\beta\mu} - 1} \to \infty$. For the first excited state,

$$\langle N_1 \rangle = \frac{1}{e^{\beta(\varepsilon_1 - \mu)} - 1} = \frac{1}{e^{\beta\varepsilon_1}\left(\frac{1}{\langle N_{\text{ground}} \rangle} + 1\right) - 1} \tag{18}$$

Since $\frac{1}{\langle N_{\text{ground}} \rangle} + 1 > 1$ and $e^{\beta\varepsilon_1} > 1$, this can never get too large. Indeed, as $\mu < 0$ there is always a gap between μ and any energy other than the ground state, so $e^{\beta(\varepsilon_1 - \mu)} > e^{\beta\varepsilon_1} > 1$. Since μ cannot get arbitrarily close to ε_i condensation cannot happen in any excited state. The ground state is special.[1]

1. Technically speaking, this is true only in equilibrium. In a laser, photons condense into an excited state. But lasers must be pumped – they are not in equilibrium.

3.1 Exact numerical solution

Now let us calculate N, and hence $\frac{\langle N_{\text{ground}}\rangle}{N}$. Bose-Einstein condensation is relevant at low temperature, where particles are non-relativistic. So consider a non-relativistic gas of monatomic bosonic atoms in a 3D box of size L. The allowed wavevectors of the system are $\vec{k}_n = \frac{\pi}{L}\vec{n}$ just like for photons or phonons, and the momenta are $\vec{p}_n = \hbar\vec{k}_n$ as always. In a non relativistic system, the energies are

$$\varepsilon_n = \frac{\vec{p}_n^2}{2m} = \frac{\hbar^2\pi^2}{2mL^2}\vec{n}^2 \tag{19}$$

We have set the ground state to $\varepsilon_0 = 0$ (rather than $\varepsilon_0 = mc^2$), since the absolute energy scale will be irrelevant. Another useful number is the gap to the first excited state

$$\varepsilon_1 = \frac{\hbar^2\pi^2}{2mL^2}(1,0,0)^2 = \frac{\hbar^2\pi^2}{2mL^2} \tag{20}$$

So that

$$\varepsilon_n = \varepsilon_1 n^2 \tag{21}$$

Then, from Eq. (16) we get

$$N = \sum_{n_x,n_y,n_z=0}^{\infty} \frac{1}{e^{\beta\varepsilon_1(n_x^2+n_y^2+n_z^2)}\left(\frac{1}{\langle N_{\text{ground}}\rangle}+1\right)-1} \tag{22}$$

This formula lets us compute N given $\langle N_{\text{ground}}\rangle$ and $\beta\varepsilon_1$. For example, if $\langle N_{\text{ground}}\rangle = 80$ and $T = 10\frac{\varepsilon_1}{k_B}$ then doing the sum numerically gives $N = 167.5$. This means with 167.5 particles at $T = 10\frac{\varepsilon_1}{k_B}$, then 80 will be in the ground state.

What we really want is to specify N and T and find $\langle N_{\text{ground}}\rangle$. Do get this function, we need to do the sum in Eq. (22) and then solve for $\langle N_{\text{ground}}\rangle$ in terms of $\beta\varepsilon_1$ and N. Unfortunately, we cannot do the sum exactly, but at least we can do it numerically. For $N = 100$ we find the numerical solution for $\frac{\langle N_{\text{ground}}\rangle}{N}$ has the form (see the Mathematica notebook on canvas)

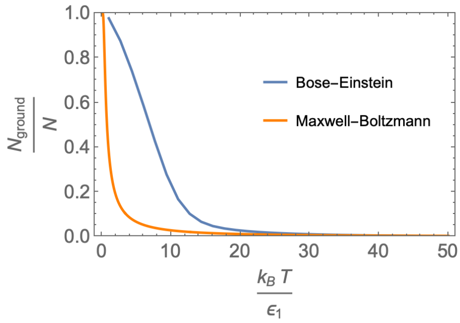


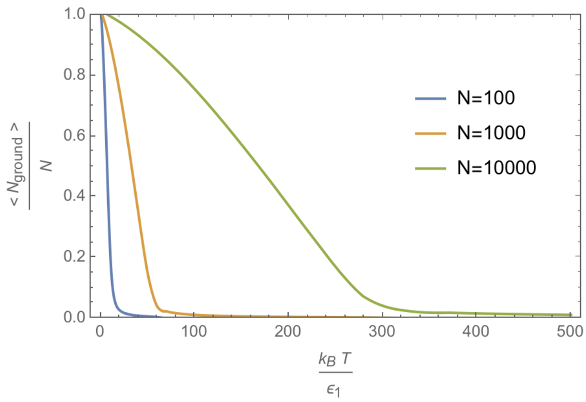
Figure 3. Exact numerical result for the ground state occupancy in a Bose system with $N = 100$.

I added to the plot the prediction using Maxwell-Boltzmann statistics. For MB statistics, we drop all the factors of ± 1. So, $\langle N_{\text{ground}}\rangle = e^{\beta\mu}$ and so Eq. (22) becomes

$$N = \sum_{n_x,n_y,n_z} \frac{1}{e^{\beta\varepsilon_1(n_x^2+n_y^2+n_z^2)}\left(\frac{1}{\langle N_{\text{ground}}\rangle}\right)} = \langle N_{\text{ground}}\rangle \sum_{n_x,n_y,n_z} e^{-\beta\varepsilon_1(n_x^2+n_y^2+n_z^2)} \tag{23}$$

These sums can be done numerically with the result is plotted alongside the BE result in Fig. 3. If we turn the sum into integrals, then $\frac{\langle N_{\text{ground}}^{\text{MB}}\rangle}{N} \approx \left(\frac{4\varepsilon_1}{\pi k_B T}\right)^{3/2}$ which looks a lot like the exact result that is plotted.

For different N the curve shifts, but looks qualitatively the same. After a little fiddling (inspired by the analytic result below), we see that if we plot the ground state occupancy as a function of $\frac{k_B T}{N^{2/3}\varepsilon_1}$ the result is essentially independent of N for the Bose-Einstein case:



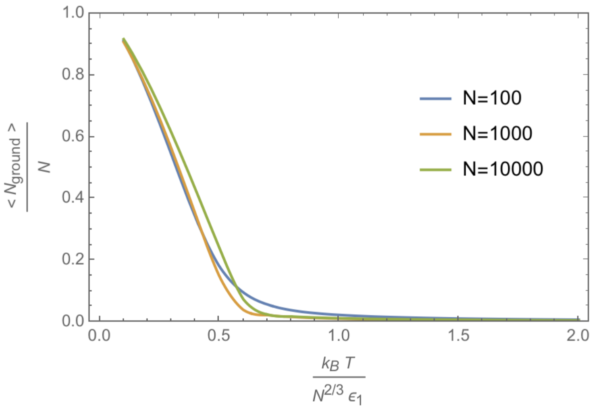


Figure 4. Ground state occupancy in a Bose system for different N as a function of T (left) and of $\frac{T}{N^{2/3}}$ (right). Pulling out a factor of $N^{2/3}$ makes the ground state occupancy essentially independent of N.

The kink in the graph, at around $T_c \approx 0.8\frac{N^{2/3}\varepsilon_1}{k_B}$ indicates the phase transition. Above this temperature, the ground state is basically empty, having only its fair share of particles. Below this temperature, $\langle N_{\text{ground}}\rangle$ starts growing linearly with T. The rescaling of our numerical result from the left to the right plot indicates that the critical temperature T_c scales like $N^{2/3}$, a result that we will next confirm analytically.

3.2 Approximate analytical solution

Having determined the exact solution numerically, let us proceed to use an analytical approach to determine some scaling relations and the transition temperature.

As with the phonon or photon gas, we first transform the sum to an integral via

$$\sum_{\vec{n}} \rightarrow \frac{1}{8}\int_0^\infty 4\pi n^2\, dn \tag{24}$$

where the $\frac{1}{8}$ comes from $\vec{n}$ being a vector of whole numbers, as in the phonon or photon case. We want to convert n to ε, which we can do using using Eq. (21), $\varepsilon = \varepsilon_1 n^2$ so

$$d\varepsilon = 2\varepsilon_1 n\, dn \tag{25}$$

So

$$\sum_{\vec{n}} \rightarrow \frac{\pi}{2}\int_0^\infty n\, n\, dn = \frac{\pi}{2}\int_0^\infty \sqrt{\frac{\varepsilon}{\varepsilon_1}}\frac{d\varepsilon}{2\varepsilon_1} = \frac{\pi}{4\varepsilon_1^{3/2}}\int_0^\infty \sqrt{\varepsilon}\, d\varepsilon \tag{26}$$

As before we write this as

$$\sum_{\vec{n}} \rightarrow \int g(\varepsilon) d\varepsilon \tag{27}$$

where

$$g(\varepsilon) = \frac{\pi}{4\varepsilon_1^{3/2}}\sqrt{\varepsilon} \tag{28}$$

is the density of states.

At this point, we would like to integrate over ε to find N

$$N = \int_0^\infty g(\varepsilon)\langle n_\varepsilon\rangle = \frac{\pi}{4\varepsilon_1^{3/2}}\int_0^\infty d\varepsilon \sqrt{\varepsilon}\frac{1}{e^{\beta(\varepsilon-\mu)}-1} \tag{29}$$

This is a little too quick, however. The problem is that converting a sum to an integral can only be justified if we do not care at all about the discreteness. For Bose-Einstein condensation we *do* care about the discreteness: the ground state, as we have seen, is qualitatively different from the other states.

Although discreteness is important for the ground sates, for the excited states, even the first excited state, there is no issue – the chemical potential μ can never approach any of their energies and so their occupancy numbers will never be unusually large. So let us proceed by taking the continuum limit for all *but* the ground state. Moreover, since $e^{-\beta\mu} \approx 1$ when $\langle N_{\text{ground}}\rangle \gtrsim 1$ which is the region of interest, we can simply set $\mu = 0$ for the excited state calculation and Eq. (29) becomes

$$\langle N_{\text{excited}}\rangle \approx \frac{\pi}{4\varepsilon_1^{3/2}} \int_{\varepsilon_1}^{\infty} d\varepsilon \sqrt{\varepsilon} \frac{1}{e^{\beta\varepsilon} - 1} = \left(\frac{\pi k_B T}{4\varepsilon_1}\right)^{3/2} \zeta_{3/2} \left[1 + \mathcal{O}\left(\sqrt{\frac{\varepsilon_1}{k_B T}}\right) + \cdots\right] \tag{30}$$

where $\zeta_{3/2} = \zeta\left(\frac{3}{2}\right) \approx 2.61$ with $\zeta(z)$ the Riemann Zeta function. The first term on the right comes from integrating from 0 to ∞. The second term on the right, scaling like $\sqrt{\frac{\varepsilon_1}{k_B T}}$ relative to the first, comes from the region $0 < \varepsilon < \varepsilon_1$. Typically $k_B T \gg \varepsilon_1$ and these corrections are small.

When does the approximation that $\mu = 0$ break down? That is $\mu \approx 0$ is a good approximation the BEC regime (low T), where the ground state is anomalously filled. It breaks down at higher T when $\langle N_{\text{ground}}\rangle$ gets small, as you can see from Eq. (14): as $\langle N_{\text{ground}}\rangle \to 0$ then $\mu \to -\infty$. We can also check this numerically, for example, by looking at the exact numerical solution for $\langle N_1\rangle$, using Eq. (18) and comparing to the $\mu = 0$ approximation, where $\langle N_1\rangle = \frac{1}{e^{\beta(\varepsilon_1 - \mu)} - 1}$:

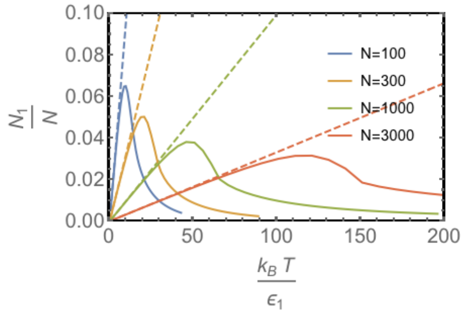


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Figure 5. Comparing $\langle N_1\rangle$ computed numerically (solid) to $\langle N_1\rangle$ in the $\mu = 0$ approximation (dashed).

So the $\mu = 0$ approximation breaks down when $\langle N_{\text{ground}}\rangle \approx 0$ and so $\langle N_{\text{excited}}\rangle \approx N$. Another way to see that the approximation is breaking down is that if we continue to apply $\mu = 0$ at higher temperatures then Eq. (30) would imply $\langle N_{\text{excited}}\rangle > N$. Indeed, setting Eq. (30) equal to N we find that our approximation breaks down when

$$\langle N_{\text{excited}}\rangle = \zeta_{3/2} \left(\frac{\pi k_B T}{4\varepsilon_1}\right)^{3/2} > N \tag{31}$$

This transition is where $\langle N_{\text{ground}}\rangle \to 0$. In other words, it occurs at the **critical temperature** where $\langle N_{\text{ground}}\rangle$, the order parameter for BEC goes from 0 to finite value. Setting $\langle N_{\text{excited}}\rangle = N$ therefore gives us a formula for the BEC critical temperature:

$$N = \zeta_{3/2} \left(\frac{\pi k_B T_c}{4\varepsilon_1}\right)^{3/2} = 2.612 \left(\frac{m k_B T_c}{8\hbar^2 \pi}\right)^{3/2} V \tag{32}$$

where with ε_1 from Eq. (20) was used. Solving for T_c gives

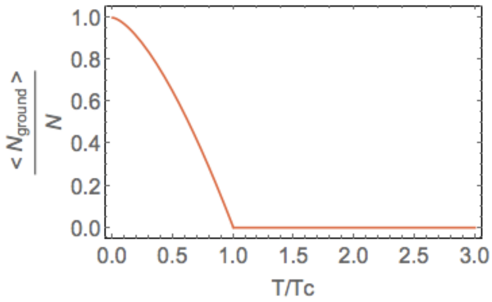
$$\boxed{T_c = \frac{4\varepsilon_1}{\pi k_B} \left(\frac{N}{\zeta_{3/2}}\right)^{2/3} = 3.31 \frac{\hbar^2}{k_B m} \left(\frac{N}{V}\right)^{2/3}} \tag{33}$$

This lets us write Eq. (30) a

$$\frac{N_{\text{excited}}}{N} = \left(\frac{T}{T_c}\right)^{3/2}, \qquad T < T_c \tag{34}$$

Thus, the fraction in the ground state is

$$\frac{\langle N_{\text{ground}}\rangle}{N} = \frac{N - \langle N_{\text{excited}}\rangle}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2} = \tag{35}$$



Comparing to our exact numerical results from Section 3.1 we find good agreement:

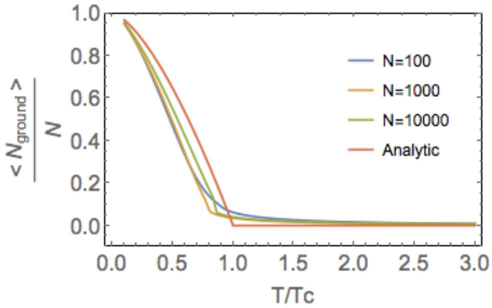


Figure 6. Comparison of the analytic result $1 - \left(\frac{T}{T_c}\right)^{3/2}$ with the exact numerical results for $N = 10^2, 10^3, 10^4$.

4 Experimental evidence

Even though the transition temperature to the BEC is well above the first excited state energy, one still needs to get a system of bosons very very cold to produce a BEC. It took 70 years from when BEC's were first conjectured theoretically (by Satyendra Nath Bose in 1924) to when technology to cool atoms was sufficiently advanced that the BEC could be produced and detected (Cornell, Weimann and Ketterle in 1995). The tricky part is that normally when you cool a gas of atoms, they solidify. A solid is not a BEC. So you need to cool the gas while keeping the density low. However, since $T_c = 3.31\frac{\hbar^2}{k_B m}\left(\frac{N}{V}\right)^{2/3}$ if you keep the density low you need very very low (ultracold) temperatures.

Since $T_c = 3.31\frac{\hbar^2}{k_B m}\left(\frac{N}{V}\right)^{2/3}$ depends inversely on the mass of the atoms lighter atoms allows the critical temperature to be higher. You might therefore think hydrogen is the easiest element to cool to see a BEC form. Unfortunately, in the mid 1990s, lasers weren't available that could operate at frequencies conducive to cooling hydrogen. Cornell and Weimann, and Ketterle, used Rubidium atoms, ^{87}Rb to form the BEC. Rubidium has a convenient set of energy levels that were will suited to the available laser cooling technologies at the time. They were able to cool around 2000 atoms using magneto-optical traps (MOTs) along with laser and evaporative cooling techniques to the nanokelvin temperature scales required for the BEC. They found that below $T_c \approx 170\,\text{nK}$ Bose-Einstein condensation can be seen. This critical temperature is in excellent agreement with the general formula $T_c = 3.31\frac{\hbar^2}{k_B m}\left(\frac{N}{V}\right)^{2/3}$ when you plug in the density they were able to achieve.

In order to see the BEC, one needs to be able to distinguish atoms in the ground state from atoms in the excited states. The basic feature that makes this possible is that the ground state atoms all have smaller wavenumbers and hence slower velocities than the other states. So if you remove the magnetic trap, the atoms will start to spread, with the ground state atoms spreading more slowly. Thus the atoms' positions after a short time indicate their initial velocities. The scientists photographed (i.e. illuminated the system with a resonant laser pulse) the system after $t = 100\,\text{ms}$ and found a high density of atoms that had not moved very far:

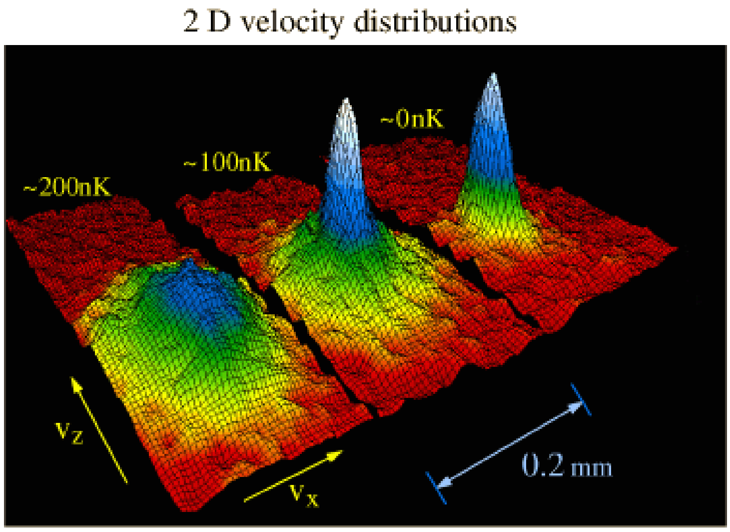


Figure 7. Observation of a Bose-Einstein condensate

In this figure, from Weimann and Cornell's group at the JILA laboratory in Boulder, we see the distribution of atoms in the BEC at different temperatures. The critical temperature is $T_c \approx 170\,\text{nK}$. Above this temperature (left), the distribution is pretty smooth, consistent with a Maxwell-Boltzmann velocity distribution. Below the critical temperature you can clearly see the higher density corresponding at an anomalously large occupancy of the ground state, consistent with expectations from a BEC. The right image shows the BEC at even lower temperature, where the occupancy is even higher. Direct observation of thermodynamic properties of the system, such as the energy density and heat capacity (particularly by Ketterle's group at), further confirmed that this system was a BEC.

Since their discovery BECs have continued to exhibit some amazing and unusual properties. Because of their coherence (all the atoms are in the *same* state), they can manifest quantum phenomena at larger scales than electrons and are in many ways more controllable than electrons. A number of groups at Harvard and MIT study BECs.

For example, Prof. Greiner (Harvard) uses lasers to localize rubidium atoms in an optical lattice. By controlling the spacing and lattice properties, he can fine tune the system, essentially choosing whatever Hamiltonian he wants. Once the rubidium atoms form a BEC, they exhibit strongly correlated behavior that can be connected to the properties of the Hamiltonian. Such an approach may lead the way to building quantum computers with longer coherence times, or to understanding what material properties might be most likely to produce room-temperature superconductors.

Another example is from Prof. Hau's lab (also at Harvard). In 1999 Prof. Hau constructed a BEC of sodium atoms. Normally, a laser tuned to a hyperfine splitting of the sodium levels would be absorbed and so sodium appears opaque to this frequency. However, Hau was able to entangle the ground and excited states of sodium using photons of a different laser in such a way that she could adjust the transparency of sodium to the first laser. The result is that she could manipulate the dispersion relation of light propagating through the sodium BEC and achieve arbitrarily small group velocities. In her first paper on the subject, she slowed light down to $17\frac{m}{s}$ with this technique. Subsequently, she was able to get light to stop completely.

The BECs produced in laboratories have carefully controlled properties, and very restricted interactions. A BEC of rubidium or sodium is unstable to condense in solid form and be held apart with some careful tricks using magnetic fields and optical traps. Helium is a noble gas that naturally has very weak interactions and will only solidify at very high pressure. At low temperature and pressure, it forms a Bose-Einstein condensate called a **superfluid**. Liquid ^{4}He is a BEC of helium atoms. Although ^{3}He is fermionic, pairs of ^{3}He atoms are bosonic, so liquid ^{3}He can be thought of as forming due to the pairing of helium atoms. Superfluids have zero viscosity. This is closely related to the bosons all being in the same state, but in this case, the state is not the zero-momentum state but one of non-zero momentum since there is a density current flowing through the fluid.

Bose Einstein condensation is also related to **superconductors**. In a superconductors, like solid mercury at $T < 4.2K$, there is no resistivity. In the BCS theory, the superconductivity is explained through the condensation of pairs of electrons called **cooper pairs**. These pairs act like bosons and form a condensate at low temperature. Because electrons are charged, so are the cooper pairs. The condensation of charged pairs screens the magnetic field in the superconductors, allowing the charged current to flow with zero resistance. Thus superconductors are much like superfluids with charged bosons instead of neutral ones.

5 Summary

Bose-Einstein condensation is a phenomenon whereby systems of bosons tend towards configurations where most of the bosons are in the identical state. Typically, you might think that at a temperature T, all the states with energy $\varepsilon \lesssim k_B T$ would be occupied. Instead, what happens is that a single state, which can have energy $\varepsilon_0 \ll k_B T$ gets an order-one fraction of the particles, while other states, even ones with $\varepsilon \ll k_B T$ as well, are hardly occupied at all.

There are various ways to understand Bose-Einstein condensation. We looked at a toy model using the canonical ensemble where we saw the condensation happen. A more powerful tool is the grand canonical ensemble. In the grand canonical ensemble, the chemical potential μ is key. The condensation happens for the state with $\varepsilon_0 \approx \mu$ for which the expected occupation $\langle n_i \rangle = \frac{1}{e^{\beta(\varepsilon_i - \mu)} - 1}$ blows up.

We found that there is a phase transition. Above a critical temperature $T_c = \frac{4\varepsilon_1}{\pi k_B}\left(\frac{N}{\zeta_{3/2}}\right)^{2/3}$, all states with $\varepsilon \lesssim k_B T$ are evenly occupied. However, below T_c, the occupation number of the ground state grows like $\frac{\langle N_{\text{ground}} \rangle}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$.

Bose-Einstein condensates include superfluids, superconductors and lasers. They were predicted in 1924 but not seen in the lab until 1995.