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Lecture 15: Refraction and Reflection

1 Refraction

When we discussed polarization, we saw that when light enters a medium with a different index of refraction, the frequency stays the same but the wavelength changes. Using that the speed of light is $v = \frac{c}{n}$ we deduced that $\lambda_1 n_1 = \lambda_2 n_2$, so that as the index of refraction goes up, the wavelength goes down. The picture looks like this

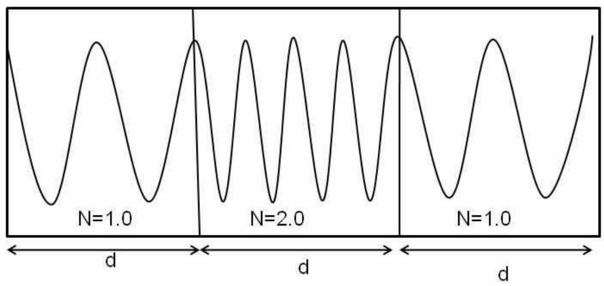


Figure 1. Plane wave entering and emerging from a medium with different index of refraction.

Now let us ask what happens when light enters a medium with a different index of refraction at an angle. Since we know the wavelength of light in the two media, we can deduce the effect with pictures. The key is to draw the plane waves as the location of the maximum field values. These crests will be straight lines, but spaced more closely together in the medium with higher index of refraction. For example, if sunlight hits ice (or water), the picture looks like this

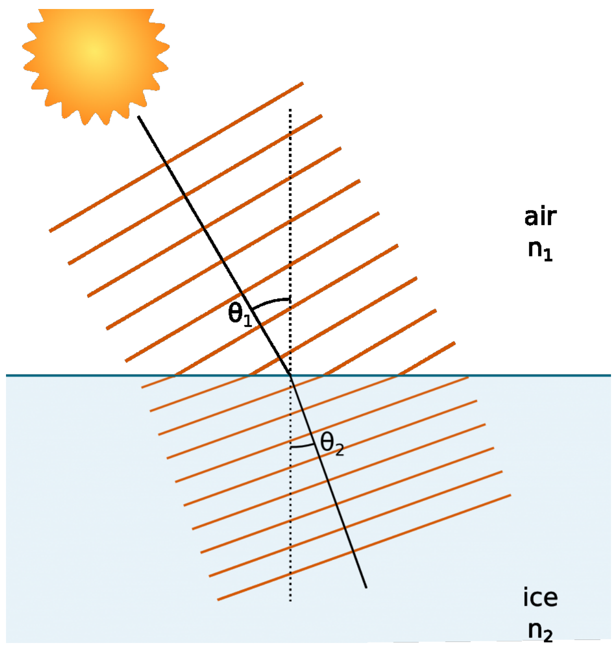


Figure 2. Matching wavefronts demonstrates refraction. Orange lines represent the crests of waves (or the maximum amplitude

The bending of light when the index of refraction changes is called **refraction**. To relate the angles θ_1 (the **angle of incidence**) to θ_2 (the **angle of refraction**) we draw a triangle

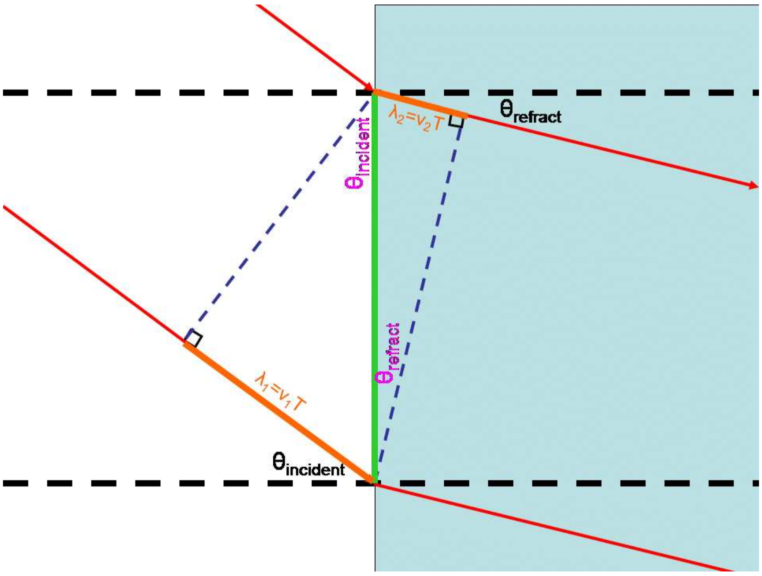


Figure 3. Light comes in from the air on the left with the left dashed blue line indicating one wavecrest with the previous wavecrest having just finished passing into the water. Thus the wavelength λ_1 in medium 1 is the solid thick orange line on the bottom left, and the wavelength in the second medium λ_2 is the thick solid orange line on the top right.

Call the distance between the places where the wave crest hits the water along the water R (the thick green vertical line in the picture). The distance between crests is $\lambda_1 = R\sin\theta_1$ in the air and $\lambda_2 = R\sin\theta_2$ in the water. Since R is the same, trigonometry and $n_1\lambda_1 = n_2\lambda_2$ imply

$$\boxed{\frac{n_1}{n_2} = \frac{\lambda_2}{\lambda_1} = \frac{\sin\theta_2}{\sin\theta_1}} \tag{1}$$

This is known as **Snell's law**.

The same logic holds for reflected waves: R is the same and λ is the same (since $n_1 = n_2$ for a reflection) therefore $\theta_1 = \theta_2$. This is usual the **law of reflection**: the angle of reflection is equal to the angle of incidence.

For a fast-to-slow interface (like air to water), the angle gets smaller (the refracted angle is less than the incident angle). For a slow-to-fast interface (like water to air), the angle gets larger. Since the angle cannot be larger than 90° while remaining in the second medium, there is a largest incident angle for refraction when $n_2 < n_1$. In equations, since the refracted angle satisfies $\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1$, we see that if $\frac{n_1}{n_2}\sin\theta_1 > 1$ there is no solution. The **critical angle** beyond which no refraction occurs is therefore

$$\boxed{\theta_c = \sin^{-1}\frac{n_2}{n_1}} \tag{2}$$

For air $n \approx 1$ and for water $n = 1.33$ so $\theta_c = 49°$. For incident angles larger than the critical angle, there is no refraction: all the light is reflected. We call this situation **total internal reflection**. Total internal reflection can only happen if $n_2 < n_1$. Thus, light can be confined to a material with higher index of refraction but not a lower one.

Total internal reflection is the principle behind fiber optics. A fiber optical cable has a solid silica core surrounding by a cladding with an index of refraction about 1% smaller. For example, the core might have $n_1 \approx 1.4475$ and the cladding $n_2 = 1.444$, so the critical angle is $\theta_c = 86°$ from normal incidence, or 4° from the direction of propagation. As long as the cable is not bent too much (typically the cable is thick enough so that this is very difficult), the light will just bounce around in the cable, never exiting, with little loss. Why might you want to send signals with visible light rather than radio waves?

In the fiber optic cable, the high index of refraction is surrounded by a lower index of refraction, or equivalently the lower wave speed medium is surrounded by higher wave speed medium. You can always remember this through Muller's analogy with the people holding hands and walking at different speeds – if slow people are surrounded by fast people, the fast will bend in to the slow. The same principle explains the SOFAR sound channel in the ocean and the sound channel in the atmosphere. Recall that for a gas, the speed of sound is $c_s = \sqrt{\gamma \frac{RT}{m}}$, thus as the temperature goes down, the speed of sound goes down. However when you go high enough to hit the ozone layer, the temperature starts increasing. This is because ozone absorbs UV light, converting it to heat. Thus speed of sound has a minimum, and there is a sound channel.

2 Boundary conditions

Snell's law holds for any polarization. It determines the direction of the transmitted fields. It does not determine the magnitude. In fact, the magnitude depends on the polarization (as we already know, since reflections are generally polarized). To determine the magnitude(s), we need the boundary conditions at the interface.

You might think $\vec{E}_1 = \vec{E}_2$ and $\vec{B}_1 = \vec{B}_2$ are the right boundary conditions. However, if charge accumulates on the boundary, as in a conductor, the fields outside and inside the material will have to be different. If current accumulates, for example through a growing number of little swirling eddies of charge, the magnetic field will be different.

How much charge or current accumulates is determined by the electric permittivity ϵ and magnetic permeability μ of a material. Recall that in a vacuum, these reduce to ϵ_0 and μ_0 and that the speed of light in the vacuum is $c = \sqrt{\epsilon_0 \mu_0}$. In a medium, we have to replace all the ϵ_0 and μ_0 factors in Maxwell's equations with ϵ and μ. So, $v = \sqrt{\mu\epsilon}$ in a material and $|\vec{B}| = \frac{1}{v}|\vec{E}|$. Moreover, since one of Maxwell's equations is $\vec{\nabla} \times \frac{\vec{B}}{\mu} = \frac{\partial}{\partial t}(\epsilon \vec{E})$, it is natural to work with the rescaled fields $\vec{D} = \epsilon \vec{E}$ and $\vec{H} = \frac{1}{\mu}\vec{B}$.

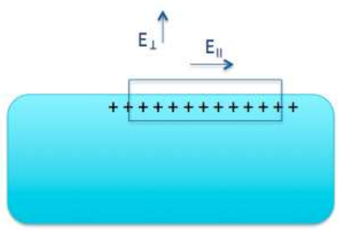


Figure 4. Charge accumulating on a boundary can affect $E_\perp$ not $E_{||}$.

To figure out the effect of the charge, we can use Gauss's law. Drawing a little pillbox around the charges as in Fig 4, we see the only the field perpendicular to the interface can be affected. Let's call this $E_\perp$. You should remember from 15b that the way electric permittivity works is that it lets you use Gauss's law as if there is no charge, provided you integrate $\vec{D} = \epsilon \vec{E}$ over the surface rather than $\vec{E}$. Therefore the boundary condition is $D_\perp^{(1)} = D_\perp^{(2)}$ which implies

$$\epsilon_1 E_\perp^{(1)} = \epsilon_2 E_\perp^{(2)} \tag{3}$$

where ϵ_1 is the dielectric constant in the medium where the transvere electric field is $E_\perp^{(1)}$ and ϵ_{21} is the dielectric constant in the medium where the transvere electric field is $E_\perp^{(2)}$.

Since the parallel electric field is unaffected by the charges, we also have

$$E_{||}^{(1)} = E_{||}^{(2)} \tag{4}$$

There are no factors of ϵ here because the accumulated charge has no effect on $E_{||}$.

For the magnetic field, the boundary conditions can be affected due to the magnetic moment of the particles in the medium, as encoded in μ. This picture looks like

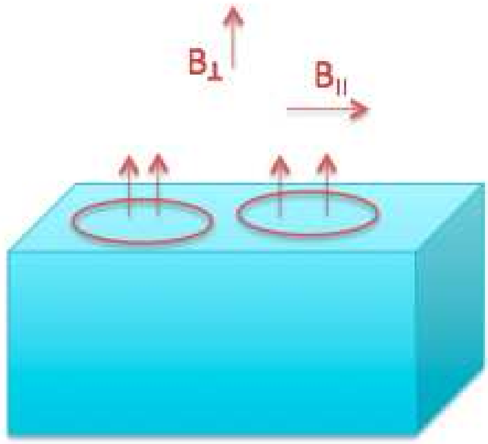


Figure 5. Current on the boundary can affect $B_{||}$ not $B_\perp$.

Since a current induces a field perpendicular to the current, only $B_{||}$ can be affected, not $B_\perp$. Thus,

$$B_\perp^{(1)} = B_\perp^{(2)} \tag{5}$$

Since the part of the magnetic field which is sensitive to an accumulated current in a material is treated using $\vec{H} = \frac{1}{\mu}\vec{B}$ the condition is then that

$$\frac{1}{\mu_1} B_{||}^{(1)} = \frac{1}{\mu_2} B_{||}^{(2)} \tag{6}$$

In summary, the boundary conditions are

$$\boxed{\begin{array}{c|c} \epsilon_1 E_\perp^{(1)} = \epsilon_2 E_\perp^{(2)} & B_\perp^{(1)} = B_\perp^{(2)} \\ \hline E_{||}^{(1)} = E_{||}^{(2)} & \frac{1}{\mu_1} B_{||}^{(1)} = \frac{1}{\mu_2} B_{||}^{(2)} \end{array}} \tag{7}$$

In the vacuum, $\mu = \mu_0 = 1.25 \times 10^{-6} \frac{H}{m}$ (Henries per meter). If μ is much larger than this, the material can acquire a large current with a small magnetic field. That is, it conducts. For example, iron has $\mu = 6.3 \times 10^{-3}$. Conductors are opaque and therefore not of much interest for the study of refraction. In most transparent materials, $\mu \approx \mu_0$. For example, water has $\mu = 1.256 \times 10^{-6} \frac{H}{m}$ which is the same as for air up to 6 decimal places. In transparent materials, the electric permittivity can vary significantly. So we will assume that $\mu \approx \mu_0$ and ϵ varies. In this case, the boxed equations reduce to $\epsilon_1 E_\perp^{(1)} = \epsilon_2 E_\perp^{(2)}$, $E_{||}^{(1)} = E_{||}^{(2)}$ and $\vec{B}^{(1)} = \vec{B}^{(2)}$.

Next, we'll solve the wave equation with these boundary conditions, much like we solved the transmission and reflection problem for waves in a string in Lecture 9 which led to the concept of impedance.

3 Normal incidence

Let's start with the case of normal incidence (perpendicular to the interface). To find out how much light is reflected, we need to work out the impedance, which determines the reflection and transmission coefficients.

For normal incidence the incident angle to the normal $\theta_1 = 0$ (see Fig. 8 below). Thus $E_\perp = B_\perp = 0$. That is, the electric and magnetic fields are both polarized in the plane of the interface to there simply is no $\perp$ component. Without loss of generality, let's take a plane wave of frequency ω moving in the $\hat{z}$ direction with $\vec{E}$ in the $\hat{y}$ direction. Then since $\vec{B} = \frac{1}{\omega}\vec{k} \times \vec{E}$, the magnetic field points in the $\hat{x}$ direction. So the incident fields are

$$\vec{E}_I = E_I \hat{y} e^{i\omega\left(t - \frac{1}{v_1}z\right)}, \quad \vec{B}_I = B_I \hat{x} e^{i\omega\left(t - \frac{1}{v_1}z\right)} \tag{8}$$

The transmitted and reflected waves are also moving normal to the surface (by Snell's law), so we can write

$$\vec{E}_T = E_T \hat{y} e^{i\omega\left(t - \frac{1}{v_1}z\right)} \quad \vec{B}_T = B_T \hat{x} e^{i\omega\left(t - \frac{1}{v_1}z\right)} \tag{9}$$

$$\vec{E}_R = E_R \hat{y} e^{i\omega\left(t + \frac{1}{v_1}z\right)} \quad \vec{B}_R = -B_R \hat{x} e^{i\omega\left(t + \frac{1}{v_1}z\right)} \tag{10}$$

with E_T, E_R, B_T and B_R the transmission and reflection coefficients to be determined. Note that we have flipped the sign on the z term in the phase since the reflected wave is moving in the $-\hat{z}$ direction (that is, $\vec{k}$ flips). Since $\omega\vec{B} = \vec{k} \times \vec{E}$, this means that $\vec{B}$ flips sign too, which explains the minus sign in Eq. (10).

Since there is no $\perp$ component the boundary condition $E_{||}^{(1)} = E_{||}^{(2)}$ implies that

$$E_I + E_R = E_T \tag{11}$$

Similarly, $\frac{1}{\mu_1} B_{||}^{(1)} = \frac{1}{\mu_2} B_{||}^{(2)}$ implies

$$\frac{1}{\mu_1} B_I - \frac{1}{\mu_1} B_R = \frac{1}{\mu_2} B_T \tag{12}$$

Since $|\vec{B}| = \frac{1}{v}|\vec{E}|$ for any plane wave, $B_I = \frac{1}{v_1} E_I$ and $B_T = \frac{1}{v_2} E_T$ and Eq. (12) becomes

$$\frac{1}{\mu_1 v_1}(E_I - E_R) = \frac{1}{\mu_2 v_2} E_T \tag{13}$$

Eqs. (11) and (13) look just very much like the equations for transmission and reflection for a wave that we studied in Lecture 9. Solving them gives

$$E_R = E_I \frac{Z_2 - Z_1}{Z_2 + Z_1}, \quad E_T = E_I \frac{2Z_2}{Z_2 + Z_1} \tag{14}$$

where

$$\boxed{Z = \mu v = \mu \frac{c}{n} = \sqrt{\frac{\mu}{\varepsilon}}} \tag{15}$$

For most materials, μ is pretty constant, so the form $Z = \mu c \frac{1}{n}$, which says $Z \propto \frac{1}{n}$, is most useful

What is the power reflected and transmitted? The incident power in a medium is given by the Poynting vector $\vec{P} = \frac{1}{\mu}\vec{E} \times \vec{B}$ times area A. So

$$|\vec{P}_I| = \frac{1}{\mu v}|\vec{E}_I|^2 A = \frac{1}{Z}|\vec{E}_I|^2 A \tag{16}$$

Thus the reflected power is

$$P_R = \frac{E_R^2}{Z_1} A = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right)^2 \frac{E_I^2}{Z_1} A = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 P_I \tag{17}$$

and the transmitted power is

$$P_T = \frac{E_T^2}{Z_2} A = \left(\frac{2Z_2}{Z_2 + Z_1}\right)^2 \frac{{E_I}^2}{Z_2} A = \frac{4 Z_1 Z_2}{(Z_1 + Z_2)^2} \frac{E_I^2}{Z_1} A = \frac{4 Z_1 Z_2}{(Z_1 + Z_2)^2} P_I \tag{18}$$

These satisfy $P_T + P_R = P_I$ as expected. Note that these equations hold for **normal incidence only**.

By the way, Eq. (15) implies that empty space has impedance of $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 376.7\Omega$. This number is very useful in broadcasting, since you want to impedance-match your antenna to air to get efficient signal transmission. Coaxial cables usually have $Z = 50\Omega$ so some circuitry is required to get the antenna impedance higher. Antennas and interference patters are the subject of Lecture 18.

4 Thin film interference

Eq. (14) is $E_R = E_I\frac{Z_2 - Z_1}{Z_2 + Z_1}$. Using $Z \propto \frac{1}{n}$, this becomes

$$\frac{E_R}{E_I} = \frac{n_1 - n_2}{n_1 + n_2} \tag{19}$$

Thus for $n_2 > n_1$ the reflected electric field has a phase flip compared to the incoming field, and for $n_1 < n_2$ there is no phase flip.

So what happens when light hits a thin film? The film will generally have $n_2 = n_{\text{film}} > n_{\text{air}} = n_1$. Thus we get a phase flip at the top surface, but no phase flip at the bottom surface. The picture looks like this

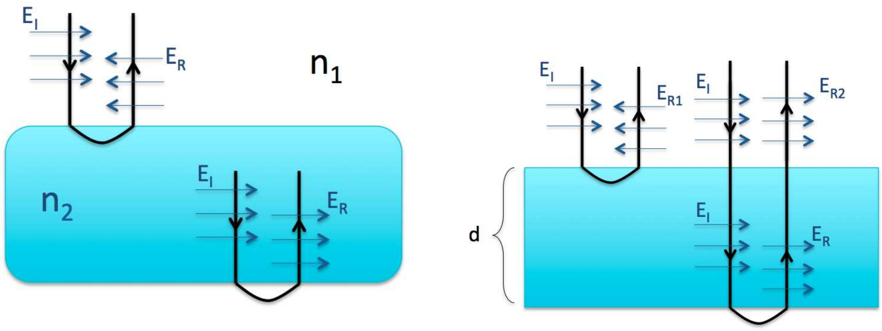


Figure 6. There is a π phase flip from reflections off the top surface, where $n_2 > n_1$, but not off the bottom surface, where $n_2 < n_1$.

So what happens if light of wavelength λ hits a film of thickness d? There will be two reflected waves, one off the top surface (A) and one off the bottom (B) which will interfere. Say the incoming electric field is $E_I\cos(kz - \omega t)$, pointing to the right. Then the wave A which reflects off the top surface ($z = 0$) will be

$$E_A = RE_I\cos(-\omega t - \pi) = -RE_I\cos(-\omega t) \tag{20}$$

with R the reflection coefficient. This now points to the left. The wave which gets through will be $ET\cos\left(\frac{2\pi}{\lambda}z - \omega t\right)$, with T the transmission coefficient. This wave then goes to the bottom suface, reflects with no phase flip, and comes back and exits with no more phase flips. So when it exits, it is back to $z = 0$ after having traversed a distance $\Delta z = 2d$. Thus it is

$$E_B = E_I T^2 R\cos\left(-\omega t + 2\pi\frac{2d}{\lambda}\right) \tag{21}$$

The total wave at $t = 0$ therefore

$$E_{\text{tot}} = E_I R\left[-1 + T^2\cos\left(\frac{4\pi d}{\lambda}\right)\right] \tag{22}$$

where λ is the wavelength in the second medium.

In the limit that $d \ll \lambda$ the two reflections will be exactly out of phase. For $T \approx 1$, there will be complete destructive interference and no reflection. But there will also be complete destructive interference whenever $\cos\left(\frac{4\pi d}{\lambda}\right) = 1$, which is when

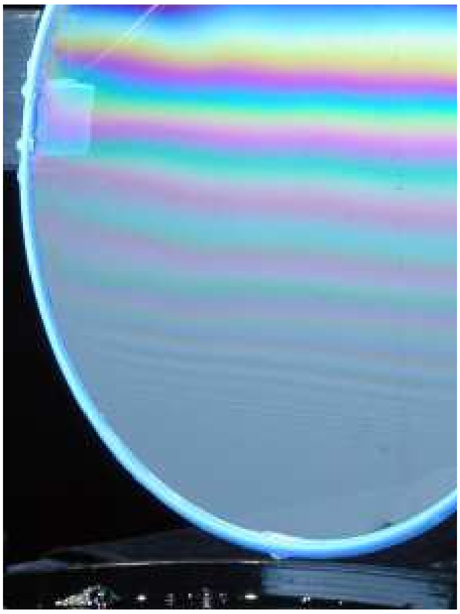
$$d = \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \frac{4\lambda}{2} \cdots \qquad (\text{complete destructive interference}) \tag{23}$$

On the other hand if the two waves are completely in phsae there will be constructive interference. This happens when $\cos\left(\frac{4\pi d}{\lambda}\right) = -1$ which is when

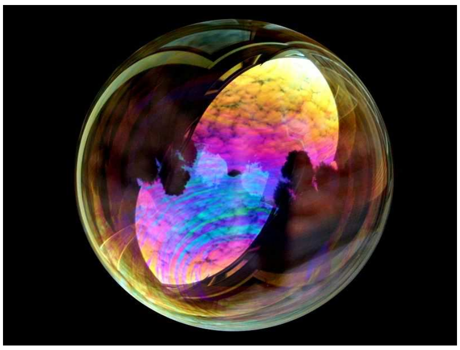
$$d = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \cdots \qquad (\text{complete constructive interference}) \tag{24}$$

If the material is much thicker than the wavelength of light, and not of completely uniform thickness, then there will be some constructive and some destructive interference and we won't see much interesting. However, if the material has a well-defined thickness which is of the same order of magnitude as the wavelength of visible light, we will see different wavelengths with different intensities. This happens in a soap film.

If we put a soap film vertically, then gravity will make it denser at the bottom. The result is the following:



Note at the very top, the film is black because there is complete distructive interference for all wavelengths. Similar patterns can be seen in soap bubbles.

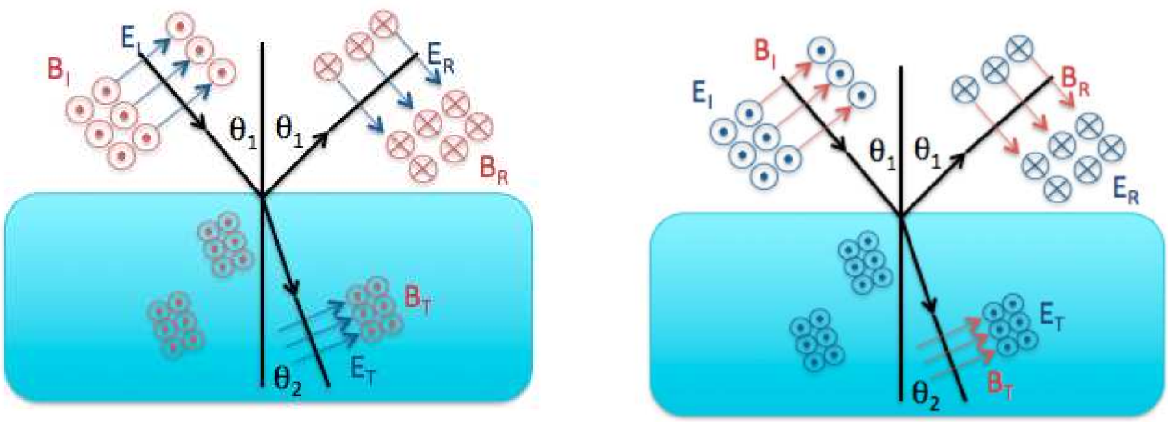


Color due to thin-film interference is known as **iridescence**. The color of many butterflies and the gorgets of hummingbirds is due to iridescence.

Figure 7. Hummingbirds an butterflies get some of their color from thin-film interference.

5 Fresnel coefficients

Now let's consider the more general case. Suppose we have a plane wave moving in the $\vec{k}$ direction towards a surface with normal vector $\vec{n}$. Thus the angle θ_1 that the wave is coming in at satisfies $\vec{n}\cdot\vec{k} = |\vec{k}||\vec{n}|\cos\theta_1$. The two vectors $\vec{k}$ and $\vec{n}$ form a plane. There are two linearly independent possibilities for the polarization: the electric field can be polarized in the $\vec{k}$-$\vec{n}$ plane of the boundary, or transverse to that plane. We call these **vertical** and **horizontal polarizations** respectively. Snell's law then tells us the directions of the transmitted and reflected electric fields. The magnetic field is always determined by the electric field through $\omega\vec{B} = \vec{k}\times\vec{E}$. What we need to solve for is the amplitude of the transmitted and reflected fields. The two cases look like:



Vertical polarization
(E in plane)

Horizontal polarization
(B in plane)

Figure 8. Two linearly independent linear polarizations are vertical and horizontal. Crosses indicate vectors point into the page, and dots that the vectors come out.

Let's start with vertical polarization. From Fig. 8 we see that $\vec{B}$ is always parallel to the surface, so $B_\perp = 0$. The electric field has $E_{||}^I = E^I\cos\theta_1$, $E_{||}^R = E^R\cos\theta_1$ and $E_{||}^T = E_T\cos\theta_2$ as well as $E_\perp^I = E^I\sin\theta_1$, $E_\perp^R = -E^R\sin\theta_1$ and $E_\perp^T = E_T\sin\theta_2$. You can check that when $\theta_1 = \theta_2 = 0$, this reduces to the normal incidence case, so we have the sines and cosines right. Note the relative sign between $E_\perp^I$ and $E_\perp^R$. You can check this sign because when $\theta = \frac{\pi}{2}$ E_I points up and E_R points down, i.e. they have opposite signs. There is is similar sign flip between B_I and B_R, as can be seen in the figure, since $\vec{B}$ always lags behind $\vec{E}$ by $\frac{\pi}{2}$.

Now we simply plug into our boundary conditions as solve. Eqs. (3) and (4) imply

$$\epsilon_1 E_I \sin\theta_1 - \epsilon_1 E_R \sin\theta_1 = \epsilon_2 E_T \sin\theta_2 \tag{25}$$

$$E_I\cos\theta_1 + E_R\cos\theta_1 = E_T\cos\theta_2 \tag{26}$$

Eq. (5) is trivially satisfied since $B_\perp = 0$. Finally, Eq. (6) implies

$$\frac{1}{\mu_1}(B_I - B_R)^{(2)} = \frac{1}{\mu_2}B_T \tag{27}$$

Using $B = \frac{1}{v}E$ the last equation becomes

$$\frac{1}{v_1\mu_1}(E_I - E_R) = \frac{1}{v_2\mu_2}E_T \tag{28}$$

Dividing Eq. (25) by Eq. (28) we get

$$\epsilon_1 v_1 \mu_1 \sin\theta_1 = \epsilon_2 v_2 \mu_2 \sin\theta_2 \tag{29}$$

Using $v = \frac{1}{\sqrt{\mu\epsilon}}$ and $n = \frac{c}{v} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$ we then get

$$n_1\sin\theta_1 = n_2\sin\theta_2 \tag{30}$$

which is Snell's law. It is reassuring that our derivation here reproduces Snell's law.

To simplify the solution it is helpful to define one parameter which depends on the angles but not on the materials

$$\alpha \equiv \frac{\cos\theta_2}{\cos\theta_1} \tag{31}$$

and another which depends only on the materials.

$$\beta = \frac{Z_1}{Z_2} = \frac{n_2\mu_1}{n_1\mu_2} = \frac{n_1\epsilon_2}{n_2\epsilon_1} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} = \sqrt{\frac{\epsilon_2\mu_1}{\epsilon_1\mu_2}} \tag{32}$$

Then the equations reduce to

$$E_I - E_R = \beta E_T \tag{33}$$

$$E_I + E_R = \alpha E_T \tag{34}$$

with solutions

$$\boxed{E_R^{\text{vert}} = \frac{\alpha-\beta}{\alpha+\beta}E_I, \quad E_T^{\text{vert}} = \frac{2}{\alpha+\beta}E_I, \quad n_1\sin\theta_1 = n_2\sin\theta_2}, \quad \text{vertical polarization} \tag{35}$$

The solution for horizontal polarization is similar,

$$\boxed{E_R^{\text{horiz}} = \frac{\alpha\beta-1}{\alpha\beta+1}E_I, \quad E_T^{\text{horiz}} = \frac{2}{\alpha\beta+1}E_I, \quad n_1\sin\theta_1 = n_2\sin\theta_2}, \quad \text{horizontal polarization} \tag{36}$$

These are known as **Fresnel coefficients**. The Fresnel coefficients tell us how much of each polarization is reflected and transmitted. Since any polarization vector can be written as a linear combination of vertical and horizontal polarizations, we can use the Fresnel coefficients to understand how any polarizations reflect.

6 Transmitted power

To interpret the Fresnel coefficients, we would like to know not just the amplitude of the wave transmitted, but the intensity of the light that passes through, or equivalently the power. Recall that the power in a plane wave in the vacuum is $P = c\epsilon_0|\vec{E}|^2$. For a plane wave in a medium, ε changes and only the component of the velocity moving into the medium is relevant, so this becomes $P = v\cos\theta\epsilon|\vec{E}|^2 = \cos\theta\sqrt{\frac{\varepsilon}{\mu}}|\vec{E}|^2 = \frac{\cos\theta}{Z}|\vec{E}|^2$. Thus the fraction of power reflected for vertical and horizontal polarizations is:

$$\frac{P_R^{\text{vert}}}{P_I^{\text{vert}}} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2, \quad \frac{P_R^{\text{horiz}}}{P_I^{\text{horiz}}} = \left(\frac{\alpha\beta-1}{\alpha\beta+1}\right)^2, \tag{37}$$

For the fraction of power transmited,

$$\frac{P_T^{\text{vert}}}{P_I^{\text{vert}}} = \frac{\cos\theta_2}{\cos\theta_1}\frac{Z_1}{Z_2}\left(\frac{2}{\alpha+\beta}\right)^2 = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2 \tag{38}$$

and

$$\frac{P_T^{\text{horiz}}}{P_I^{\text{horiz}}} = \frac{\cos\theta_2}{\cos\theta_1}\frac{Z_1}{Z_2}\left(\frac{2}{\alpha\beta+1}\right)^2 = \alpha\beta\left(\frac{2}{\alpha\beta+1}\right)^2 \tag{39}$$

One can check that for either polarization, $P_T + P_R = P_I$ and that these equations agree with Eqs. (17) and (18) for normal incidence $\left(\alpha = 1, \beta = \frac{Z_1}{Z_2}\right)$.

For example, the air-glass interface has $\beta = 1.5$. Then the transmitted and reflected power in vertical and horizontal polarizations are

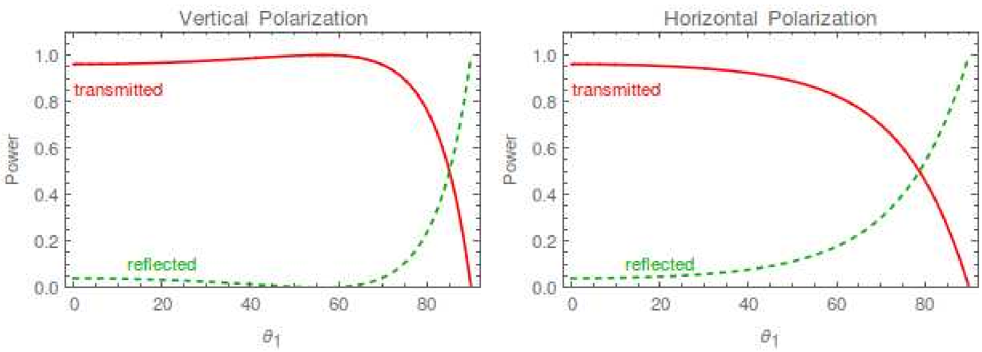


Figure 9. Transmitted and reflected power as a function of incident angle for the two polarizations.

We can also plot the ratio of power reflected in vertical to horizontal polarizations:

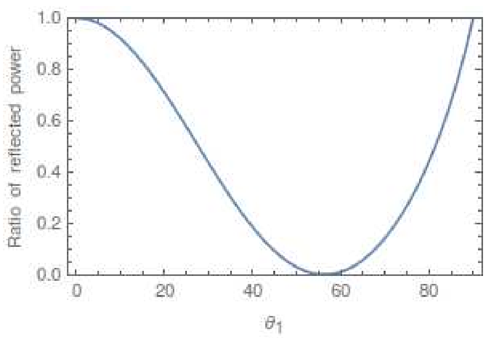


Figure 10. Ratio $P_R^{\text{vert}}/P_R^{\text{horiz}}$ of power reflected vertically polarized to horizontally polarized light, .

From these plots we see there is an angle where the vertical polarization exactly vanishes. This angle is called **Brewster's angle**, θ_B. We can see from the plots that for the glass-air interface $\theta_B \approx 56°$. At this angle, the reflected light is completely polarized.

What is the general formula for θ_B? From Eq. (37) we see that $P_R^{\text{vert}} = 0$ when $\alpha = \beta$. That is

$$\frac{\cos\theta_2}{\cos\theta_B} = \sqrt{\frac{\epsilon_2\mu_1}{\epsilon_1\mu_2}} \tag{40}$$

For most materials $\mu_1 \sim \mu_2 \approx \mu_0$. Then we can use $n = \sqrt{\mu\epsilon}$ to get $\frac{\cos\theta_2}{\cos\theta_B} = \frac{n_2}{n_1}$. Using also $n_1\sin\theta_B = n_2\sin\theta_2$ we can solve for θ_B giving

$$\boxed{\tan\theta_B = \frac{n_2}{n_1}} \tag{41}$$

For the air-water interface, $\theta_B = \tan^{-1}1.5 = 56.3°$. One can see from Figure 10 that one does not have to be exactly at this angle to have little reflected vertical polarization. Angles close to θ_B work almost as well.

What is going on physically at Brewster's angle? We know that Snell's law $n_1\sin\theta_1 = n_2\sin\theta_2$ is always satisfied. At Brewster's angle, when $\theta_1 = \theta_B$, we found also $n_1\cos\theta_2 = n_2\cos\theta_1$. Dividing these two equations gives $\cos\theta_1\sin\theta_1 = \cos\theta_2\sin\theta_2$, or equivalently

$$\sin(2\theta_1) = \sin(2\theta_2) \quad \text{when}\ \theta_1 = \theta_B \tag{42}$$

Now, by definition θ_1 and θ_2 are both between 0 and $\frac{\pi}{2}$, so $2\theta_1$ and $2\theta_2$ are between 0 and π. Thus, $\sin(2\theta_1) = \sin(2\theta_2)$ has two solutions. Either $\theta_1 = \theta_2$, which corresponds to $n_1 = n_2$ so the light just passes through, or $\frac{\pi}{2} - 2\theta_1 = 2\theta_2 - \frac{\pi}{2}$ (see Fig 11, left) which simplifies to

$$\theta_1 + \theta_2 = \frac{\pi}{2} \tag{43}$$

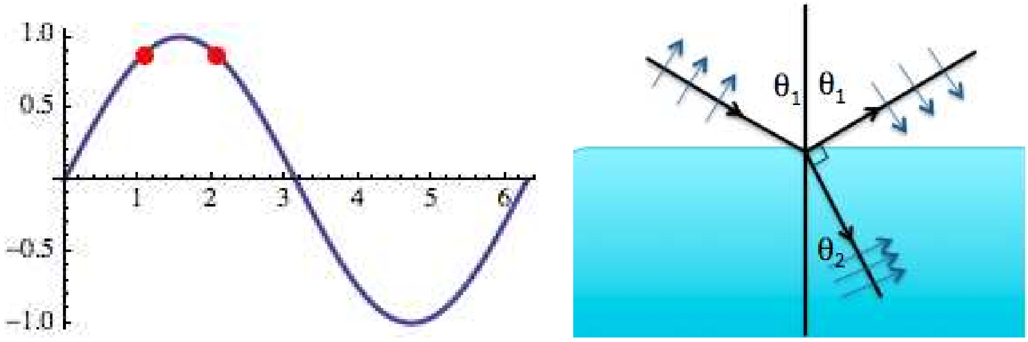


Figure 11. The Brewster's angle happens when $\sin(2\theta_1) = \sin(2\theta_2)$ which corresponds to $\frac{\pi}{2} - 2\theta_1 = 2\theta_2 - \frac{\pi}{2}$ as can be seen in the left figure. This means that the transmitted and reflected waves are perpendicular. Thus the reflected vertically polarized light (shown on the right) cannot be produced by the motion of particles in the surface. Hence the reflected vertically polarized light vanishes at Brewster's angle.

Working out the geometry, as in Fig. 11, we see that the transmitted and reflected waves are perpendicular at Brewster's angle. Since the reflected wave has to be produced by the motion of particles in the surface we can understand why there is no reflection at Brewster's angle: particles moving in the surface can only produce light polarized transverse to their direction of motion. In Lecture 17, we will understand in more detail how accelerating changes produce electromagnetic fields and waves.

The famous Harvard dropout Edwin Land started a corporation called Polaroid that made its first fortune with polarizing sunglasses. These glasses remove glare from reflection by removing the horizontally polarized light. He made his second fortune was made with the Polaroid camera. In 1973 Land donated the money for the construction of the Science Center (which some people say looks like a camera...).