

Lecture 13:

Electromagnetic waves

1 Light as waves

In previous classes (15a, high school physics), you learned to think about light as rays. Remember tracing light as lines through a curved lens and seeing whether the image was inverted or right side up and so on? This ray-tracing business is known as **geometrical optics**. It applies in the limit that the wavelength of light is much smaller than the objects which are bending and moving the light around. We're not going to cover geometrical optics much in 15c (except for what you do in lab). Approaching light as waves will allow us to understand topics of polarization, interference, and diffraction which you may not have seen in a physics class before.

Electromagnetic waves can have any wavelength. Electromagnetic waves with wavelengths in the visible spectrum ($\lambda \sim 10^{-6} \text{ m}$) are called **light**. Red light has a longer wavelength than blue light. Longer wavelengths than red go into infrared, then microwaves, then radio waves. Smaller wavelengths than blue are ultraviolet, then χ rays than γ rays. Here is a diagram of the spectrum:

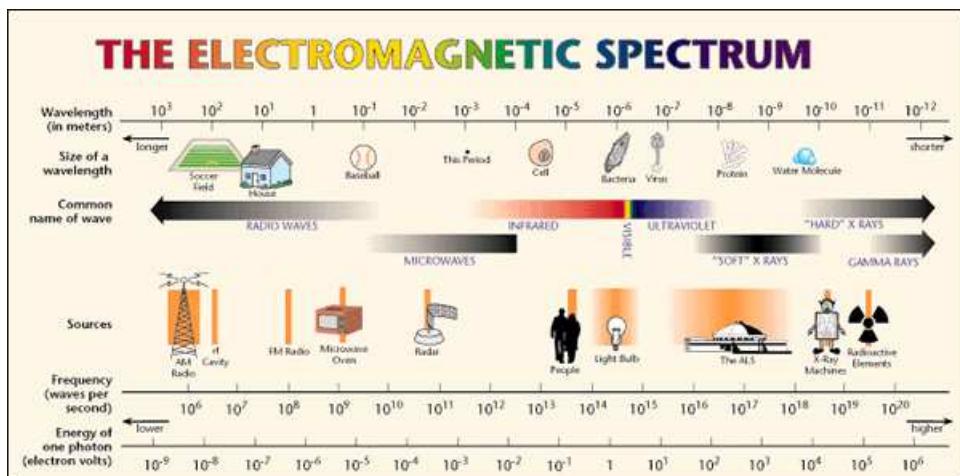


Figure 1. Electromagnetic spectrum

It is helpful to go back and forth between wavelengths λ , frequency $\nu = \frac{c}{\lambda}$, and energy $E = h\nu$, where $h = 6.6 \times 10^{-34} \text{ J} \cdot \text{s}$ is Planck's constant. The fact that energy and frequency are linearly related is a shocking consequence of quantum mechanics. In fact, although light is wavelike, it is made up of particles called **photons**. A photon of frequency ν has energy $E = h\nu$. The intensity of light (recall intensity is power/area, $I = \frac{P}{A}$) is also the number density n of photons per unit volume, times their rate (the speed of light c), times their energy: $I = nch\nu$. Thus, given a frequency of light, the intensity of light at that frequency tells you how many photons are present. It's usually a lot. For example, light from the sun has intensity $I = 1.2 \frac{\text{W}}{\text{m}^2}$. Assuming this peaks in the visible ($\sim 500\text{nm}$), we get $n = \frac{I\lambda}{c^2 h} = 10^{22} \text{ m}^{-3}$. This is roughly Avogadro's number per cubic meter.

We use the words soft and hard to mean lower energy and higher energy respectively. That's because higher energy photons just plow through things, like a bullet, while lower energy photons are easy to deflect. Let's start by briefly going through the spectrum from hard to soft (small λ to big λ), discussing some of the relevant physics.

Very energetic photons are called **gamma rays**. Gamma rays are a type of radiation coming from radioactive decay. Their high energy makes these rays very dangerous. Gamma rays can also be seen coming in from outer space with energies of 10^{19} eV or wavelengths as small as 10^{-25} m!! These are called **cosmic rays**.

Typical atomic spacings in a solid are of order 1 Angstrom = 10^{-10} m. This is not surprisingly the wavelength of x-rays; their wavelength, also of order 10^{-9} m, can probe these solids. Hard (meaning shorter wavelength/larger energy) x-rays, as used in x-ray machines scatter off of electrons in solids. They usually scatter inelastically, knocking the electron out of its orbit and ionizing the atom. Thus they are called **ionizing radiation**. Ionizing radiation can be dangerous. The more dense a solid, the more electrons are there to scatter off of, so the x-rays go through less before scattering. That's why x-rays machines can distinguish dense materials like metal or bone from less dense ones like skin. Soft x-rays have wavelengths of multiple atomic spacings. These can be used to probe how a solid is organized through the wave nature of light. These softer x-rays are used in x-ray crystallography.

There is an important relationship between temperature and the wavelength where the spectrum of a blackbody peaks: $\lambda_{\max} T = 2.898 \text{ mm K}$. This is known as **Wein's displacement law**. We will discuss it more in Lecture 15. For $T = 5778 \text{ K}$ the surface temperature of the sun, $\lambda_{\max} = 501 \text{ nm}$, which is yellow light. Thus sunlight peaks as yellow.

Visible light has wavelengths $\lambda \sim 10^{-7} \text{ m}$ which are a few orders of magnitude longer than typical atomic spacings. However the energy of visible light, \sim few electron volts, is the same as the spacing between energy levels in atoms. More precisely, the key unit is a Rydberg,

$$1 \text{ Ry} = \frac{1}{2} m_e c^2 \alpha^2 = 13 \text{ eV} \quad (1)$$

with m_e the electron mass, and $\alpha \approx \frac{1}{137}$ the fine-structure constant. Visible light has energy of a few Rydbergs. Materials have different colors because of resonances which excite electronic transitions in materials absorbing the light. This is a quantum phenomenon. We will discuss color in Lecture 15.

Infrared light ($\lambda = 10^{-4} \text{ m} \sim 10^{-6} \text{ m}$) has lower energy than visible light. It cannot excite electron transitions in molecules, but it can excite vibrational modes. These excitations are broader (lower Q) and more sensitive to the thermal motion. They do not usually appear as sharp resonances. By Wein's displacement law, a blackbody at temperature 310 K (like mammals) has a peak wavelength of $\lambda_{\max} = 10^{-5} \text{ m}$ which is in the infrared. That's why infrared cameras which see heat maps in the dark. Some snakes are also sensitive to infrared radiation for the same reason.

Microwaves ($\lambda = 10^{-3} \text{ m} \sim 10^{-1} \text{ m}$) are longer wavelength/lower energy than infrared. They can excite rotational modes of water, which is how microwave ovens work. Microwaves are also used in full body airport scanners: they can see skin under clothing because they see the water in the skin! (Alternative airport scanners use ionizing x-rays at low intensity and look at the photons which backscatter).

Longer wavelength radiation are generically called radio waves. They are very low energy, non-ionizing, and not dangerous. The higher frequency the radio waves, the more information can be carried (see Lecture 12). For example, you need about 10 kHz bandwidth for audio and 5 MHz bandwidth for television. AM radio has frequencies around 1 MHz ($\lambda \sim 100 \text{ m}$), FM radio around 100 MHz ($\lambda \sim 1 \text{ m}$). Cell phones and wireless data transfer are usually done in GHz bands ($\lambda \sim 1 \text{ cm}$). 3G technology has around 1.7 GHz carrier frequencies, and 4G is 2.5 GHz. Although higher frequencies than this could transmit more data, we don't want cell phones to transmit in the microwave region (for obvious reasons!).

2 Wave equation from Maxwell's equations

With no charges around ($\rho = \vec{J} = 0$), as in empty space, Maxwell's equations are

$$\boxed{\vec{\nabla} \cdot \vec{E} = 0 \quad \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \vec{E}} \quad (2)$$

Taking a time derivative of the 4th equation gives

$$\mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\vec{E} = \frac{\partial}{\partial t}[\vec{\nabla} \times \vec{B}] = \vec{\nabla} \times \frac{\partial}{\partial t}\vec{B} = -\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) \quad (3)$$

where the 2nd Maxwell equation was used in the last step. Now we can use the identity

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - (\vec{A} \cdot \vec{B})\vec{C} \quad (4)$$

For $\vec{A} = \vec{B} = \vec{\nabla}$ and $\vec{C} = \vec{E}$ gives

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} = -\vec{\nabla}^2 \vec{E} \quad (5)$$

where the 1st Maxwell equation was used in the last step. We therefore have

$$\mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\vec{E} = \vec{\nabla}^2 \vec{E} \quad (6)$$

or more simply

$$\boxed{\left[\frac{\partial^2}{\partial t^2} - c^2 \vec{\nabla}^2 \right] \vec{E}(\vec{x}, t) = 0} \quad (7)$$

where $c \equiv \frac{1}{\sqrt{\mu_0\epsilon_0}}$ is the **speed of light**. Thus the electric field satisfies the wave equation. With similar manipulations, you can also show that

$$\boxed{\left[\frac{\partial^2}{\partial t^2} - c^2 \vec{\nabla}^2 \right] \vec{B}(\vec{x}, t) = 0} \quad (8)$$

Thus, the magnetic field must also satisfy the wave equation. Moreover, since the second Maxwell equation is $\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$ the solutions for the electric and magnetic fields are not independent. Thus we call the **electromagnetic waves**. We will next see how to solve these coupled equations.

3 Plane waves

Recall from Lecture 10 that for a sound wave, satisfying

$$\boxed{\left[\frac{\partial^2}{\partial t^2} - v^2 \vec{\nabla}^2 \right] p(\vec{x}, t) = 0} \quad (9)$$

with $p(\vec{x}, t)$ the pressure at position \vec{x} and time t , plane waves looked like

$$p(x, y, z, t) = p_0 \cos(\vec{k} \cdot \vec{x} - \omega t + \phi) \quad (10)$$

Here, \vec{k} is the wavevector. The wave equation implies $\omega = v|\vec{k}|$. These waves are constant in the plane perpendicular to \vec{k} . For example, if $\vec{k} = (0, 0, k_z)$ then

$$p(x, y, z, t) = p_0 \cos(k_z z - \omega t + \phi) \quad (11)$$

with $\omega = v k_z$. For a given frequency ω , there can be plane waves with any wavevector \vec{k} satisfying $|\vec{k}| = \frac{\omega}{v}$ and any phase ϕ . Any wave can be written as a sum over plane waves

$$p(\vec{x}, t) = \int dk_x dk_y dk_z \tilde{p}(\vec{k}) e^{i(\vec{k} \cdot \vec{x} - \omega t)} \quad (12)$$

with \tilde{p} in general complex. Or for real p we can write

$$p(\vec{x}, t) = \int d^3k \tilde{p}_s(\vec{k}) \sin(\vec{k} \cdot \vec{x} - \omega t) + \int d^3k \tilde{p}_c(\vec{k}) \cos(\vec{k} \cdot \vec{x} - \omega t) \quad (13)$$

where d^3k is shorthand for $dk_x dk_y dk_z$.

For electromagnetic waves, the plane wave solutions to Eq. (7) are similar

$$\vec{E}(\vec{x}, t) = \operatorname{Re} \left[\vec{E}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)} \right] \quad (14)$$

with

$$\omega = c |\vec{k}| \quad (15)$$

We usually leave the taking of the real part implicit and work with complex electric and magnetic fields for simplicity. We can always take the real part at the end of the day.

A critical difference with sound or other waves is that the coefficient of the wave \vec{E}_0 is itself a vector. Thus, in addition to a direction \vec{k} the wave is moving, there is a direction \vec{E}_0 describing which way the electric field is pointing. We call \vec{E}_0 the **polarization vector for the electric field**. As with any plane wave, a plane wave electric field is constant in the direction perpendicular to \vec{k} . For example, if $\vec{k} = (0, 0, k_z)$ then \vec{E} is independent of x and y at a fixed z and t .

Now, the Maxwell's equation $\vec{\nabla} \cdot \vec{E}(x, t) = 0$ implies that for a plane wave solution with wavevector \vec{k} that

$$0 = \vec{\nabla} \cdot \vec{E}(\vec{x}, t) = \vec{\nabla} \cdot \vec{E}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)} = i(\vec{k} \cdot \vec{E}_0) e^{i(\vec{k} \cdot \vec{x} - \omega t)} \quad (16)$$

The only way this can be satisfied at all times is if

$$\boxed{\vec{k} \cdot \vec{E}_0 = 0} \quad (17)$$

So the **polarization is orthogonal to the direction of propagation of the plane wave**.

Since the magnetic field satisfies the same equations $\left[\frac{\partial^2}{\partial t^2} - c^2 \vec{\nabla}^2 \right] \vec{B}(\vec{x}, t) = 0$ and $\vec{\nabla} \cdot \vec{B} = 0$, its plane wave solutions are similar

$$\vec{B}(\vec{x}, t) = \vec{B}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}, \quad \vec{k} \cdot \vec{B}_0 = 0 \quad (18)$$

Finally, we can use that $\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{B}$. For an electric plane wave

$$\vec{\nabla} \times \vec{E} = \vec{\nabla} \times \vec{E}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)} = i(\vec{k} \times \vec{E}_0) e^{i(\vec{k} \cdot \vec{x} - \omega t)} \quad (19)$$

For a magnetic plane wave

$$-\frac{\partial}{\partial t} \vec{B}(\vec{x}, t) = -\frac{\partial}{\partial t} \vec{B}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)} = i\omega \vec{B}_0 e^{i(\vec{k} \cdot \vec{x} + \omega t)} \quad (20)$$

Setting $\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{B}$ we see that **the wavevectors \vec{k} must be the same for \vec{E} and \vec{B}** . We also find

$$\omega \vec{B}_0 = \vec{k} \times \vec{E}_0 \quad (21)$$

Since the vector $\vec{A} \times \vec{B}$ is perpendicular to both \vec{A} and \vec{B} , Eq. (21) implies that \vec{B}_0 is perpendicular to both \vec{k} (which we already knew) and \vec{E}_0 . Also, since $\omega = c |\vec{k}|$ by Eq. (15) we have

$$|\omega \vec{B}_0| = |\vec{k} \times \vec{E}_0| = |\vec{k}| |\vec{E}_0| = \frac{\omega}{c} |\vec{E}_0| \quad (22)$$

where we have used that \vec{k} is perpendicular to \vec{E}_0 to write $|\vec{k} \times \vec{E}_0| = |\vec{k}| |\vec{E}_0|$. Thus we find

$$\boxed{|\vec{E}_0| = c |\vec{B}_0|} \quad (23)$$

So the amplitude for the electric and magnetic fields in a plane wave are related by the speed of light.

Keep in mind that $|\vec{E}| = c |\vec{B}|$ is a special condition for plane waves. It certainly does not hold for most solutions to Maxwell's equations. For example, we can certainly have pure electric fields (or pure magnetic fields). These satisfy Maxwell's equations in empty space but do not have $|\vec{E}| = c |\vec{B}|$. However, because any solution can be written as a sum of plane waves, by Fourier's theorem, discussing plane waves where $|\vec{E}| = c |\vec{B}|$ will be very productive.

In summary, for plane waves, we have found for a given wavevector \vec{k} we have

$$\vec{k} \perp \vec{E}, \quad \vec{k} \perp \vec{B}, \quad \vec{E} \perp \vec{B}, \quad |\vec{E}_0| = c|\vec{B}_0| \quad (24)$$

We can draw a plane wave like this

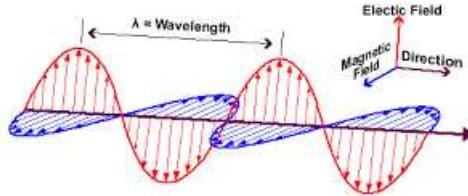


Figure 2. Propagation of a linearly polarized plane wave where \vec{k} points to the right. In this wave, the \vec{E} field always points in vertically, the \vec{B} field horizontally, and their magnitudes vary with position in sync.

This picture illustrates some things but hides others. The \vec{E} field is constant in magnitude and direction in the entire plane perpendicular to \vec{k} . It points, at every position, in the same direction. Taking the real part (to find the actual electric field):

$$\text{Re}(\vec{E}) = \vec{E}_0 \cos(\vec{k} \cdot \vec{x} - \omega t + \phi) \quad (25)$$

we see that \vec{E} varies in time and in the \vec{k} direction. At any point, the \vec{B} field is exactly perpendicular to the \vec{E} field. The magnitude of the \vec{B} field is always $|\vec{B}_0| = \frac{1}{c}|\vec{E}_0|$ at all points at all times. \vec{B} and \vec{E} are always in phase.

Finally, note that for a given \vec{k} and phase, there are two orthogonal directions that \vec{E} can point. Once \vec{E} is fixed, \vec{B} is fixed too. These are the two linearly independent linear polarizations of light. Any other solution to the wave equation in free space can be written as a sum of plane waves of these two polarizations.

4 Energy and power

In physics 15b you derived that the energy density in the electromagnetic field is

$$\mathcal{E} = \frac{1}{2}\epsilon_0|\vec{E}|^2 + \frac{1}{2}\frac{1}{\mu_0}|\vec{B}|^2 \quad (26)$$

This holds for *any type of electromagnetic field*, not just plane waves. I'm not going to re-derive this equation, but we can quickly check that it makes sense – we saw for other waves that the energy is proportional to the square of the amplitude, as it is in this case. The factors of ϵ_0 and μ_0 can then be put in by dimensional analysis.

What we are more interested in here is the energy transported by a wave. We can write

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial t} &= \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \frac{1}{\mu_0} \vec{B} \cdot \frac{\partial \vec{B}}{\partial t} \\ &= \frac{1}{\mu_0} (\vec{E} \cdot (\vec{\nabla} \times \vec{B}) - \vec{B} \cdot (\vec{\nabla} \times \vec{E})) \\ &= \frac{1}{\mu_0} \vec{\nabla} \cdot (\vec{B} \times \vec{E}) \end{aligned}$$

The total energy in a volume V is $E_V = \int_V \mathcal{E}$ so that

$$\frac{dE_V}{dt} = \frac{1}{\mu_0} \int_V \vec{\nabla} \cdot (\vec{B} \times \vec{E}) = \frac{1}{\mu_0} \oint (\vec{B} \times \vec{E}) d \cdot \vec{A} \quad (27)$$

where \vec{A} is the area element on the surface of V . This is just the divergence theorem. This says that

$$\vec{S} \equiv -\frac{dE_V}{dt \text{ Area}} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{\text{watts}}{\text{meter}^2} \quad (28)$$

This is called the **Poynting vector**. It gives the power flowing *into* a region (the minus sign in Eq. (28) makes it *into* the region rather than *out of* the region). Note that since $\vec{E} \times \vec{B} \propto \vec{k}$ for a plane wave, the Poynting vector points in the direction of the wave motion. So power flows in the \vec{k} direction. The Poynting vector measures the **intensity of a wave**.

Using $\omega \vec{B}_0 = \vec{k} \times \vec{E}_0$, for a plane wave the magnitude of the intensity is

$$I = |\vec{S}| = \frac{1}{\mu_0 c} \vec{E}^2 = c \epsilon_0 \vec{E}^2 \quad (29)$$

So the intensity is given by the square of the electric (or magnetic) fields.

In a sense, light is a transverse wave since \vec{E} and \vec{B} are transverse to the direction \vec{k} of the wave propagation. On other hand, light acts like a longitudinal wave in that it carries momentum. To see that, consider an electric charge which the wave hits. The electric field will push the charge in the \vec{E} direction. Then once the charge is moving, the magnetic field will push the charge in the $\vec{v} \times \vec{B}$ direction. Since $\vec{v} \propto \vec{E}$, this is the $\vec{E} \times \vec{B} \propto \vec{k}$ direction. That is, the combination of the electric and magnetic fields has the net effect of pushing the charge along the direction \vec{k} that the wave is going. Thus the charge ends up starting to spin in a little helix going in the \vec{k} direction. The effect is that light exerts pressure on matter. The pressure is called **radiation pressure**. The radiation pressure is given by the Poynting vector divided by c :

$$\vec{p} = \frac{1}{c} \vec{S} \quad (30)$$

By conservation of momentum, the light wave must also carry momentum.

So light carries momentum in the direction of $\vec{S} \propto \vec{k}$. Since light is made up of photons, each photon must also carry momentum proportional to \vec{k} . We give the proportionality constant a name, Planck's constant h :

$$\vec{p} = h \vec{k} \quad (31)$$

In other words, knowing that light is made up of photons is enough to derive that the momentum of the photons is proportional to \vec{k} . Since \vec{k} is the Fourier conjugate of \vec{x} , if a particle is well localized in \vec{x} , it must be poorly localized in \vec{p} (as we learned from our study of wavepackets). This is the origin of the Heisenberg uncertainty principle. Of course, you probably can't make any sense out of what it means for an electron to be "well-localized" in x . To understand this concept requires quantum mechanics.

Of course, mostly when electromagnetic radiation passes through an object, not much of the momentum is transferred. If the radiation is completely absorbed, as visible light is on black paper, the momentum is completely transferred.

By the way, Kepler postulated radiation pressure way back in 1619 to explain why the tails of comets are always pointing away from the sun.

5 Historical note (optional)

Since the electric and magnetic fields satisfy the wave equation, it is natural to ask what are the little oscillators which are strung together to allow the waves to propagate? One case is that they might be air molecules. This guess is quickly disproved since the propagation is independent of the density of air. In particular, electromagnetic waves propagate perfectly well in a vacuum. Thus the medium is something new which we call the **aether**.

Maxwell proposed a model of the aether based on gears and wheels. These gears have to be able to move very rapidly to support high frequencies. A number of other models were proposed in the 19th century. The aether has never been discovered. It is also not needed since Maxwell's equations correctly predict all of the phenomenology associated with electromagnetic waves without the need for an aether.

Aether theories were disproved by the famous Michelson-Morley experiment. This experiment showed that the speed of light is the same in different reference frames through **interferometry**. We will cover interferometry later, and in lab. The idea is that there were some microscopic system whose oscillations generated electromagnetic waves, then the speed of the waves should depend on the speed of the aether. For example, if you put the shive wave machine on a truck, then the speed that the waves propagate in the machine as observed by someone on the ground depend on the speed of the truck. If you put light on a truck, it still goes at the speed of light.

We now understand that the speed of light is the same in all frames follows from a symmetry principle. That is, we start with an axiom of frame-independence, called **Lorentz invariance**, and use it to derive the constancy of the speed of light. Lorentz invariance is the generalization of spatial rotations to include time. For example, with one spatial direction x and time t , a Lorentz transformation is a boost (change of frames) which has the form

$$x' = x \cosh \beta + ct \sinh \beta \quad (32)$$

$$t' = t \cosh \beta + \frac{x}{c} \sinh \beta \quad (33)$$

where

$$\sinh \beta \equiv \frac{e^x - e^{-x}}{x} \quad \cosh \beta \equiv \frac{e^x + e^{-x}}{x} \quad (34)$$

These satisfy

$$\cosh^2 \beta - \sinh^2 \beta = 1 \quad (35)$$

Then

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t} \frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t} \frac{\partial}{\partial x'} = \cos \beta \frac{\partial}{\partial t'} + c \sinh \beta \frac{\partial}{\partial x'} \quad (36)$$

$$\frac{\partial}{\partial x} = \frac{\partial t'}{\partial x} \frac{\partial}{\partial t'} + \frac{\partial x'}{\partial x} \frac{\partial}{\partial x'} = \frac{1}{c} \sinh \beta \frac{\partial}{\partial t'} + \cosh \beta \frac{\partial}{\partial x'} \quad (37)$$

so

$$\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} = [\cosh^2 \beta - \sinh^2 \beta] \frac{\partial^2}{\partial t'^2} + c^2 [\cosh^2 \beta - \sinh^2 \beta] \frac{\partial^2}{\partial x'^2} \quad (38)$$

$$= \frac{\partial^2}{\partial t'^2} - c^2 \frac{\partial^2}{\partial x'^2} \quad (39)$$

So the wave equation is Lorentz invariant. That is, it is the same equation, with the same speed c in any frame.