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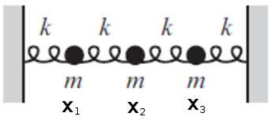
Lecture 6: Waves in strings and air

1 Introduction

In Lecture 4, we derived the wave equation for two systems. First, by stringing together masses and springs and taking the continuum limit, we found



$$\Rightarrow \quad \left[\frac{\partial^2}{\partial t^2} - v^2 \frac{\partial^2}{\partial x^2}\right] A(x,t) = 0 \tag{1}$$

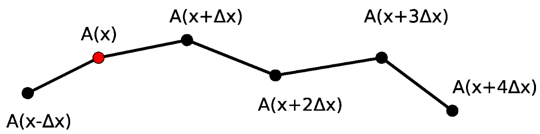
where $A(x,t)$ is the displacement from equilibrium of the mass at position x. These are longitudinal waves. In this equation, for waves in a solid, the wave speed is

$$v = \sqrt{\frac{E}{\mu}} \tag{2}$$

where E is the elastic modulus and μ is the density per unit length. Now we consider two more cases: transverse oscillations on a string and longitudinal motion of air molecules (sound waves).

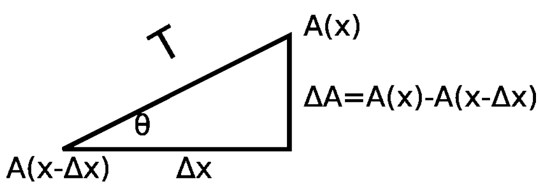
2 Transverse oscillations

Consider a string of tension T. We define the amplitude of the string at a point x at time t as $A(x,t)$. In this section, we'll sometimes write $A(x,t)$ just as $A(x)$ to avoid clutter. Let us treat the string as a bunch of massless test probes connected by a elastic strings. Then we can draw the picture as



What is the force acting on the test mass at position x (in red)?

First, consider the *downward* component of the force pulling on the test mass at x from the mass to the left (at $x - \Delta x$). We can draw a triangle:



The force is given by

$$F_{\text{downwards, from left mass}} = T\sin\theta = T\frac{\Delta A}{\sqrt{\Delta A^2 + \Delta x^2}} \tag{3}$$

If the system is close to equilibrium, then the slope will be small. That is, $\Delta A \ll \Delta x$. In this case, we can approximate $\sqrt{\Delta A^2 + \Delta x^2} \approx \Delta x$ and so

$$F_{\text{downwards, from left mass}} = T\frac{\Delta A}{\Delta x} = T\frac{A(x) - A(x - \Delta x)}{\Delta x} = T\frac{\partial A}{\partial x} \tag{4}$$

where we have taken $\Delta x \to 0$ in the last step turning the difference into a derivative.

Similarly, the *downward* force from the mass on the right is

$$F_{\text{downwards, from right mass}} = T\frac{A(x) - A(x+\Delta x)}{\Delta x} = -T\frac{\partial A(x+\Delta x)}{\partial x} \tag{5}$$

Thus,

$$F_{\text{total downwards}} = -T\left[\frac{\partial A(x+\Delta x)}{\partial x} - \frac{\partial A}{\partial x}\right] \tag{6}$$

Now we use $F = m a$, where $a = -\frac{\partial^2 A}{\partial t^2}$ is the downward acceleration. So we should have

$$F_{\text{total downwards}} = -m\frac{\partial^2 A}{\partial t^2} = -\mu \Delta x \frac{\partial^2 A}{\partial t^2} \tag{7}$$

Plugging this into Eq. (6) we find

$$\frac{\partial^2 A}{\partial t^2} = \frac{T}{\mu}\left[\frac{\frac{\partial A(x+\Delta x)}{\partial x} - \frac{\partial A}{\partial x}}{\Delta x}\right] = \frac{T}{\mu}\frac{\partial^2 A}{\partial x^2} \tag{8}$$

Thus,

$$\boxed{\left[\frac{\partial^2}{\partial t^2} - v^2\frac{\partial^2}{\partial x^2}\right]A(x,t) = 0 \quad \text{with} \quad v = \sqrt{\frac{T}{\mu}}} \tag{9}$$

So the wave equation is again satisfied with a wave speed $v = \sqrt{\frac{T}{\mu}}$.

Note that the acceleration is due to a **difference of forces**. The force pulling up from the right has to be different from the force pulling down from the left to get an acceleration. Each force is proportional to a first derivative, thus the acceleration is proportional to a second derivative.

3 Sound waves

Waves in air are just like waves in a solid: the air molecules are like little masses and the forces between them act like springs. Thus we have already derived the wave equation. What's left is to think about what's actually going on when a wave propagates through the air.

Sound waves are longitudinal density waves, which look like

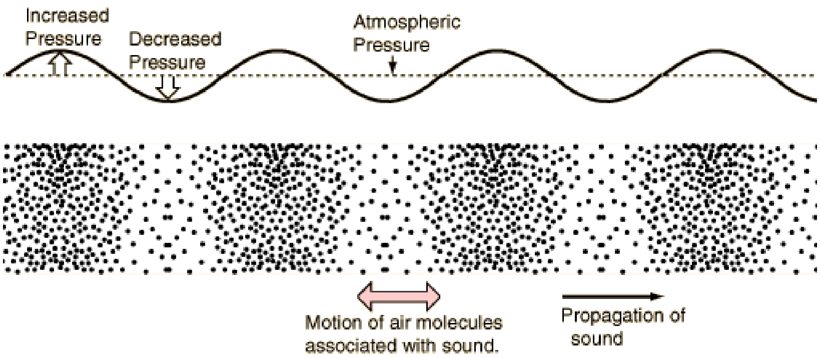


Figure 1. Visualization of sound waves

What is the amplitude for a sound wave? As always, the amplitude $A(x,t)$ measures the displacement from equilibrium. In fact, in a sound wave, each individual molecule is just oscillating back and forth around an equilibrium position, and the wave appears as a collective phenomenon among these moving molecules. This is easiest to see in an animation. Try the animation on this web page http://www.acs.psu.edu/drussell/demos/waves/wavemotion.html under longitudinal waves. In this animation, $A(x,t)$ is the displacement from equilibrium at time t of the red dot whose equilibrium position is at x. It has the form $A(x,t) = A_0\cos(kx - \omega t + \phi_0)$ for some overall amplitude A_0 and some phase ϕ_0. From the point of view of the molecules strung together, this system is identical to the masses and springs strung together that we discussed in lecture 4. So the derivation of the wave equation for a gas is identical.

In a snapshot of the wave in Fig. 1, it's hard to see which molecules are at their equilibrium position and which are not. That is, it's hard to see $A(x,t)$. Instead, we see the density of the gas $\rho(x,t)$. In fact the two are related. Looking the animations for a long time, you can see that in fact there is a close relation between the amplitude (displacement from equilibrium) $A(x,t)$ and the density $\rho(x,t)$. As with any oscillator, the molecules move fastest as they pass through their equilibrium going left or right, and stop when they are farthest from equilibrium. Now note that when the molecules (red dots on the web animation) are moving fastest to the right, they are in the most dense region, and when they moving fastest to the left, they are in the least dense region of the gas. That is, the maximal velocity in the $+x$ direction corresponds to the maximal density and the minimal velocity in the $+x$ direction to the minimal density. Therefore density agrees with velocity: $\rho \propto \rho_0 + \frac{dA}{dt}$. In other words, if the displacement is $A(x,t) = A_0\cos(kx - \omega t)$ then the density is $\rho(x,t) = \rho_0 + (\Delta\rho)\sin(kx - \omega t)$. Another way to say this is that the density lags behind the amplitude by 90°.

3.1 Speed of sound in a gas

Let us consider the case where the sound wave is excited by a large membrane like a drum or a speaker. If we are interested in wavelengths much less than the size of the membrane, and much larger than the distance between air molecules, then waves in air become exactly like waves in a solid or waves on a string. We simply have to divide by the unit area:

$$\frac{\mu}{A}\frac{\partial^2 A}{\partial t^2} = \frac{T}{A}\frac{\partial^2 A}{\partial x^2} \tag{10}$$

Now, $\frac{\mu}{A}$ is the mass per unit length per unit area, also known as mass per unit volume or density ρ. Also, $\frac{T}{A}$ is the force per unit area or the pressure, so we get

$$\rho\frac{\partial^2 A}{\partial t^2} = p\frac{\partial^2 A}{\partial x^2} \tag{11}$$

Thus $v = \sqrt{\frac{p}{\rho}}$ for a gas. It turns out that this is only correct at constant temperature.

At constant temperature, the gas doesn't heat up. You may remember the ideal gas law from chemistry:

$$pV = nRT \tag{12}$$

where V is the volume, n is the number of molecules, R is the ideal gas constant and T is the temperature. Dividing by V and using $\rho = \frac{n}{V}m$ with m the molecular weight of the gas molecules, we get

$$p = \rho\frac{RT}{m} \tag{13}$$

This means that

$$\left(\frac{dp}{d\rho}\right)_T = \frac{RT}{m} = \frac{p}{\rho} \tag{14}$$

where the subscript T means "constant temperature". Thus we would have $v = \sqrt{\left(\frac{\partial p}{\partial\rho}\right)_T}$ for a gas which could not heat up. There is unfortunately no such gas.

The correct velocity for a wave in air is

$$v = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_S} \tag{15}$$

where the subscript S means "constant entropy". It should be constant entropy since when a wave passes through some air, it leaves the air in the same state it started in, without increasing the entropy. If you take physical chemistry (physics 181 or chem 161), you can study these constrained partial derivatives to death. I will just summarize the important result

$$\left(\frac{\partial p}{\partial \rho}\right)_S = \gamma\frac{p}{\rho} \tag{16}$$

where

$$\gamma = \frac{C_P}{C_V} \tag{17}$$

where C_P is the specific heat at constant pressure and C_V is the specific heat at constant volume.

A more useful form of γ is

$$\gamma = \frac{f+2}{f} \tag{18}$$

where f is the number of decrease of freedom in the gas. For a monatomic gas, like argon, the only degrees of freedom are from translations. For the x, y and z directions, we get $f=3$. So

$$\gamma = \frac{3+2}{3} = \frac{5}{3} = 1.67 \quad \text{(monatomic gas)} \tag{19}$$

For a diatomic gas, like N_2 or O_2 (which is mostly what air is), both atoms can move, so we would get $f=6$, however, if they rotate around the bond axis, the molecule is unchanged, so in fact $f=5$. You can think of 5 as 3 translations, one rotation and one vibration along the bond axis. Thus

$$\gamma = \frac{5+2}{5} = \frac{7}{5} = 1.4 \quad \text{(diaatomic gas like air)} \tag{20}$$

To match to the notation for waves in solids we sometimes define a **bulk modulus**

$$B \equiv \gamma p \tag{21}$$

Then the speed of sound in air is

$$c_s = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{B}{\rho}} \tag{22}$$

Note that B and c_s are properties of the gas, not the wave. All waves have the same velocity in the same type of air.

Another useful formula is that, using the ideal gas law,

$$c_s = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma \frac{RT}{m}} \tag{23}$$

This tells us that the speed of sound only depends on the temperature of a gas, not on its density or pressure separately. It also tells us the speed of sound is different in two gases with the same temperature but different molecular masses m.

You may also recall from chemistry that the root-mean-square velocity of gas is determined by its temperature: $v_{\text{rms}} = \sqrt{\frac{3RT}{m}}$. Again, this is something you will show in a physical chemistry class. So

$$c_s = \sqrt{\frac{\gamma}{3}} v_{\text{rms}} \tag{24}$$

Thus the speed of sound is proportional to, but not greater than, the speed of the molecules in the gas. This makes sense – how could it be sound travel faster than the molecules transmitting it?

3.2 Summary

For sound waves, the amplitude of the wave $A(x, t)$ is the displacement from x of the molecule whose equilibrium position is at x. The density of the wave $\rho(x, t)$ also oscillates and lags in phase behind the amplitude by a quarter wavelength, $\frac{\pi}{2}$. Sound waves satisfy the wave equation with a sound speed

$$v = \sqrt{\gamma \frac{p}{\rho}} = \sqrt{\gamma \frac{RT}{m}} \tag{25}$$

where p is the average pressure, ρ the average density and T the average temperature. Also,

$$\gamma = \frac{C_P}{C_V} = \frac{f+2}{f} \tag{26}$$

where C_P is the specific heat at constant pressure and C_V is the specific heat at constant volume and f is the number of degrees of freedom of the gas molecules. For a monotonic gas like Ar, $f=3$ and $v=\sqrt{1.67\frac{p}{\rho}}$. For a diatomic gas like N_2 or O_2, $f=5$ and $v=\sqrt{1.4\frac{p}{\rho}}$.

4 Standing waves

Now lets talk about standing wave solutions in more detail. Again, we consider the wave equation

$$\left[\frac{\partial^2}{\partial t^2}-v^2\frac{\partial^2}{\partial x^2}\right]A(x,t)=0 \tag{27}$$

and we would like solutions of fixed frequency ω. These are solutions which are periodic in time. We can write the general such solution as a sum of terms of the form

$$A(x,t)=A_0\sin(kx+\phi_1)\sin(\omega t+\phi_2) \tag{28}$$

In this solution, A_0 is the **amplitude** and k the **wavenumber**. The frequency determined from the wavenumber through the dispersion relation

$$\omega=vk \tag{29}$$

There are two phases ϕ_1 and ϕ_2. Instead of using phases, we could write the general solution as

$$A(x,t)=A_0\sin(kx)\sin(\omega t)+A_1\sin(kx)\cos(\omega t)+A_2\cos(kx)\cos(\omega t)+A_3\cos(kx)\sin(\omega t) \tag{30}$$

The two forms are equivalent and we will go back and forth between them as convenient.

Consider first the case where one of the boundary conditions is that the string is fixed at $x=0$. That is

$$\boxed{A(0,t)=0} \tag{31}$$

This is known as a fixed, closed, or **Dirichlet** boundary condition. If there were a $A_3\cos(kx)\sin(\omega t)$ component, then the $x=0$ point would oscillate as $x(0,t)=A_3\sin(\omega t)$ meaning it is not fixed. Thus $A_3=0$. Similarly, $A_2=0$. Thus the general solution with $A(0,t)=0$ is

$$A(x,t)=A_0\sin(kx)\sin(\omega t+\phi) \tag{32}$$

If we fix the other end of the string at $x=L$ then we must have $\sin(kL)=0$ which implies

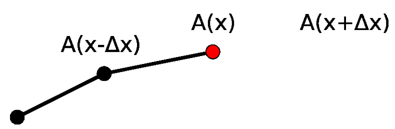
$$k=\frac{\pi}{L}n, \quad n=1,2,3,\cdots \tag{33}$$

This tells us which frequencies can be produced

$$\boxed{\omega_n=vk_n=v\frac{\pi}{L}n, \quad n=1,2,3,\cdots} \quad \text{both ends fixed} \tag{34}$$

This is the spectrum for 2 Dirichlet boundary conditions.

Next, consider having the end at $x=0$ fixed but the end at $x=L$ free. To figure out what happens if the end is free, we have to go back to our picture



Now there is no line connected $A(x)$ to $A(x+\Delta x)$. Then, $F_{\text{right}}=0$ and Eq. (6) becomes

$$F_{\text{total}}=F_{\text{left}}+F_{\text{right}}=T\frac{\partial A}{\partial x} \tag{35}$$

So, Eq. (8) becomes

$$\frac{T}{\mu}\frac{\partial A}{\partial x}=\Delta x\,\frac{\partial^2 A}{\partial t^2} \tag{36}$$

In this case if we take Δx to 0 we see that $\frac{\partial A}{\partial x} \to 0$. Thus a free end must satisfy

$$\boxed{\frac{\partial A(L,t)}{\partial x} = 0} \tag{37}$$

This is known as a free, open, or **Neumann** boundary condition.

Now using the $x=0$ fixed solution, Eq. (32), the Neumann condition at $x=L$ implies

$$0 = \frac{\partial A(L,t)}{\partial x} = k A_0 \cos(kL) \sin(\omega t + \phi) \tag{38}$$

For this to hold at all times, $\cos(kL)$ must be at a zero of the cosine curve. Now, $\cos(x) = 0$ when $x = \left(n + \frac{1}{2}\right)\pi$. Thus,

$$\boxed{\omega_n = v \frac{n + \frac{1}{2}}{L} \pi, \quad n = 0, 1, 2, 3}, \quad \text{one fixed end,} \;\; \text{one free end} \tag{39}$$

This solution says that the lowest frequency is

$$\nu_0 = \frac{\omega_1}{2\pi} = \frac{1}{2} v \frac{\frac{1}{2}}{L} = \frac{1}{4} \frac{v}{L} \tag{40}$$

the next frequency up is

$$\nu_1 = \frac{1}{2} v \frac{1 + \frac{1}{2}}{L} = \frac{3}{4} \frac{\nu}{L} = 3\nu_0 \tag{41}$$

and so on. Thus the even harmonics are missing!! This has dramatic consequences for instruments like the trumpet and the clarinet.

Finally, if $x = 0$ is free, we must have $A(x,t) = A_0 \cos(kx) \sin(\omega t + \phi)$. Then, if $x = L$ is also free, we find $\sin(kL) = 0$ which implies

$$\boxed{\omega_n = v \frac{n}{L} \pi, \quad n = 0, 1, 2, 3}, \quad \text{both ends free} \tag{42}$$

The only difference between both free ends solution and both fixed end solution is that for free ends $n = 0$ is allowed. However, the $n = 0$ solution is $A(x,t) = \text{const}$ which has $k = \omega = 0$ thus it is not physically interesting.

Here are the lowest harmonics with different boundary conditions

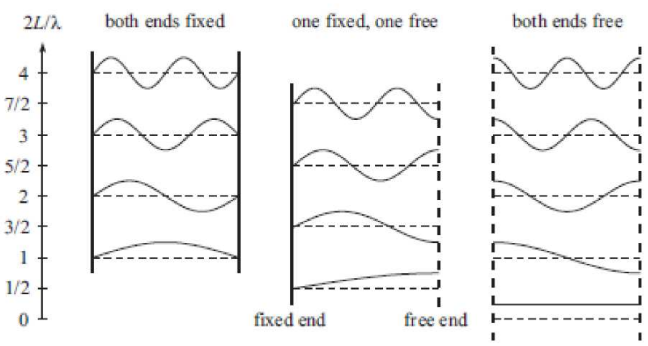


Figure 2. Frequencies allowed for different boundary conditions

If the fundamental note (lowest frequency) is ν, then we find

	lowest	next	second	third
both fixed	ν	2ν	3ν	4ν
one fixed, one free	ν	3ν	5ν	7ν
both free	ν	2ν	3ν	4ν

(43)

What are the implications for this? Well for one thing different instruments correspond to different boundary conditions. String instruments have both ends fixed. Woodwinds and brass have one end open. A flute has both ends free. The absence of the even harmonics is one of the reasons that clarinets tend to sound eerie. Many of the complications in the designs of brass instruments help restore the even harmonics. This is explained well in Rick Heller's book. See for example this figure from page 317: (don't worry too much about understanding this picture now – it will make more sense after the next lecture, on music, but it naturally falls in to this lecture):

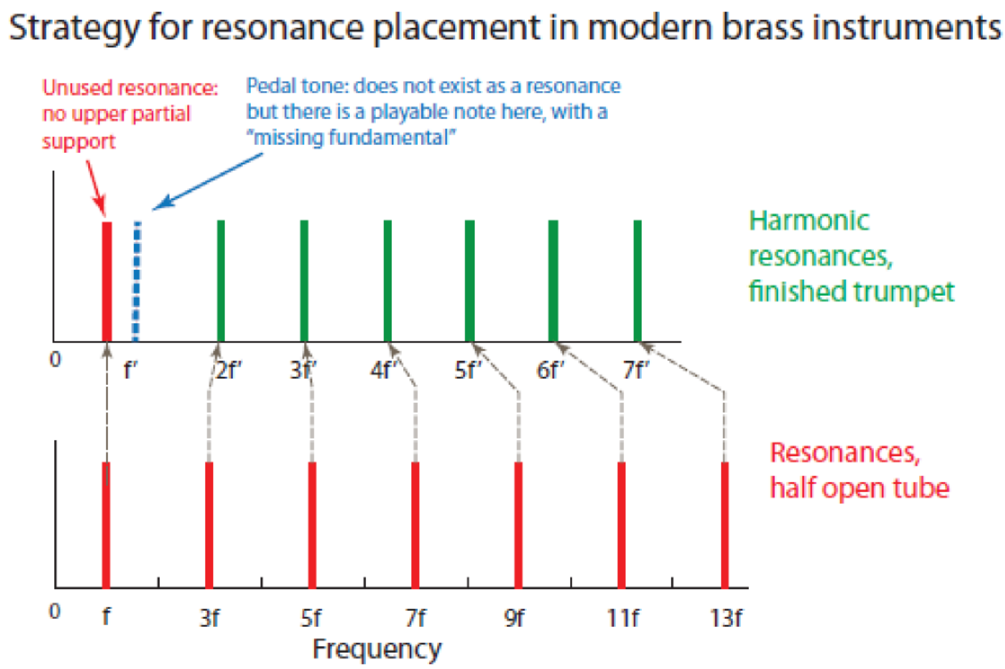


Figure 16.6: The shifting of the even-numbered resonances of a half open tube (bottom, red), based on a fundamental f, to a set of equally spaced odd and even harmonic resonances based on a fundamental "pedal tone" f', a frequency which does not exist as a resonance. This is accomplished with a bell and mouthpiece. The pedal tone is playable, due to the harmonic support it gets, but is normally not used. The lowest, quarter-wave resonance (red) is not part of the harmonic series (green) and is also unused.

5 Helmholtz resonators (optional)

An important object in the physics of sound is the resonance chamber or **Helmholtz resonator**. Helmholtz resonators are hollow cavities with small openings, like a bottle or a violin body. They work because the volume of the air in the body cannot change, thus pushing down on the air in the neck forces the air in the body to push back with essentially a linear restoring force, like a spring. The setup is like this

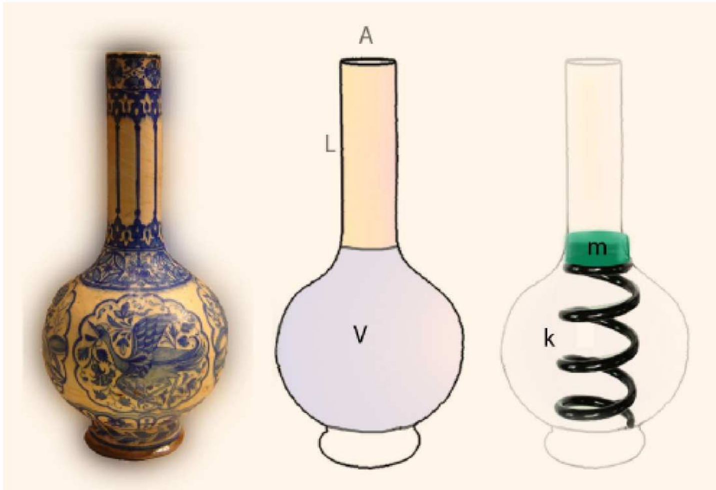


Figure 3. In a Helmholtz resonator, the air in the neck acts like a mass and the air in the base acts like a spring. Figure taken from Fig 13.1 of Heller.

To work out the resonant frequency for a Helmholtz resonator we can use $\omega = \sqrt{\frac{k}{m}}$. We can extract the spring constant k from $F = -k\Delta x$. For pressure, $F = A \cdot dp$, where A is the area, in this case the cross sectional area of the neck. Now, $\rho = \frac{m}{V}$ so

$$d\rho = \frac{d}{dV}\left(\frac{m}{V}\right)dV = -\frac{m}{V^2}dV = -\rho\frac{dV}{V} \tag{44}$$

Also using Eq. (16), $\frac{dp}{d\rho} = \gamma\frac{p}{\rho}$ for sound waves, we have

$$dp = \frac{dp}{d\rho}d\rho = \gamma\frac{p}{\rho}d\rho = -\gamma\frac{p}{V}dV \tag{45}$$

Now, $dV = A\Delta x$ and so

$$F = A \cdot dp = -\gamma A^2\frac{p}{V}\Delta x \tag{46}$$

Thus

$$k = \gamma A^2\frac{p}{V} = A^2 c_s^2\frac{\rho}{V} \tag{47}$$

The mass on which the spring acts is the air in the neck. It has mass $m = \rho A L$, thus

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{A^2 c_s^2 \rho / V}{\rho A L}} = c_s\sqrt{\frac{A}{VL}} \tag{48}$$

Thus Helmholtz resonators resonate at a single frequency

$$\boxed{\nu = \frac{c_s}{2\pi}\sqrt{\frac{A}{VL}}} \tag{49}$$

where A is the area of the opening, L is the length of the neck, and V is the volume of the cavity.

For example, consider a 10 cm wide jar with a 10 cm long neck. Using $v = 343\,\frac{m}{s}$, $A = 1\text{cm}^2$, $L = 1\text{cm}$, $V = 1L = 1000\text{cm}^3$, we find $\nu = 172\,\text{Hz}$. The associated wavelength in air is $\lambda = \frac{c_s}{\nu} = 2m$. Note that the wavelength of sound in the resonator is much larger than the size of the resonator.

Since Helmholtz resonators have only one frequency, they have no harmonics (no overtones). However, they can have low Q values. Indeed, if you blow on a bottle, you see that the sound does not resonate for long at all. This is good, if you are building an instrument, since you want all the audible frequencies to resonate. On a violin, the vibrations are produced on the strings, transmitted to the wooden body of the violin through the bridge (the part of the violin which connects the strings to the body). The body then vibrates, exciting the air in the body which emits sound through the holes. I can't describe the function of the body of a violin better than Heller. Here's his description [Heller, p. 267]

> Helmholtz resonators can be used as transducers, turning mechanical energy into sound energy. A prime example is the violin. The violin body is basically a box containing air, with the f-holes opening to the outside. It functions deliberately as a Helmholtz resonator, enhancing the low frequency response of the violin, giving it much of its richness of tone...
>
> The violin body's broad Helmholtz resonance peaks around 300 Hz. No doubt the shape thin but long holes serve to increase air friction and thus lower the Q of the Helmholtz mode, spreading the resonance over a broader frequency range. This props up the transduction of string vibrations into sound down to the frequency of the open D string [$\nu \sim 300$Hz].