Matthew Schwartz

Lecture 5: Fourier series

1 Fourier series

When N oscillators are strung together in a series, the amplitude of that string can be described by a function $A(x,t)$ which satisfies the **wave equation**:

$$\left[\frac{\partial^2}{\partial t^2} - v^2\frac{\partial^2}{\partial x^2}\right]A(x,t) = 0 \tag{1}$$

We saw that electromagnetic fields satisfy this same equation with $v = c$ the speed of light.

We found normal mode solutions of the form

$$A(x,t) = A_0\cos\left(\frac{\omega}{v}(x \pm vt) + \phi\right) \tag{2}$$

for any ω which are **traveling waves**. Solutions of the form

$$A(x,t) = A_0\cos(kx)\cos(\omega t) \tag{3}$$

with $\omega^2 = v^2k^2$ are called **standing waves**. Whether traveling waves or standing waves are relevant depends on the boundary condition.

More generally, we found traveling wave solutions could come from any function $f(x+vt)$:

$$\left[\frac{\partial^2}{\partial t^2} - v^2\frac{\partial^2}{\partial x^2}\right]f(x+vt) = 0 \tag{4}$$

Similarly $f(x-vt)$ is a solution. Functions $f(x-vt)$ are right-moving traveling waves and functions $f(x+vt)$ are left-moving traveling waves.

Now, since any vector can be written as a sum of eigenvectors, any solution can be written as a sum of normal modes. This is true both in the discrete case and in the continuum case. Thus we must be able to write

$$f(x+vt) = \sum_k a_k\cos(kx)\cos(\omega t) + b_k\sin(kx)\cos(\omega t) + c_k\cos(kx)\sin(\omega t) + d_k\sin(kx)\sin(\omega t) \tag{5}$$

where the sum is over wavenumbers k. In particular, at $t=0$ any function can be written as

$$f(x) = \sum_k a_k\cos(kx) + b_k\sin(kx) \tag{6}$$

We have just proved Fourier's theorem!

(Ok, we haven't really proven it, we just assumed the result from linear algebra about a finite system applies also in the continuum limit. The actual proof requires certain properties about the smoothness of $f(x)$ to hold. But we are physicists not mathematicians, so let's just say we proved it.)

2 Fourier's theorem

Fourier's theorem states that any square-integrable function[1] $f(x)$ which is periodic on the interval $0 < x \leqslant L$ (meaning $f(x+L) = f(x)$) can be written as

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{2\pi n}{L}x\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{2\pi n}{L}x\right)} \tag{7}$$

1. A function is square-integrable if $\int_0^L dx f(x)^2$ exists.

with

$$a_0 = \frac{1}{L}\int_0^L dx\, f(x) \tag{8}$$

$$a_n = \frac{2}{L}\int_0^L dx\, f(x)\cos\left(\frac{2\pi n}{L}x\right) \tag{9}$$

$$b_n = \frac{2}{L}\int_0^L dx\, f(x)\sin\left(\frac{2\pi n}{L}x\right) \tag{10}$$

This decomposition is known as a **Fourier series**. Fourier series are useful for periodic functions or functions on a fixed interval L (like a string). One can do a similar analysis for non-periodic functions or functions on an infinite interval ($L \to \infty$) in which case the decomposition is known as a Fourier transform. We will study Fourier series first.

It is easy to verify these formulas for a_n and b_n. For a_0, we just integrate $f(x)$. Since $\cos\left(\frac{2\pi}{L}nx\right)$ goes through n cycles of the complete cosine curve as x goes from 0 to L, we have

$$\int_0^L dx\cos\left(\frac{2\pi n}{L}x\right) = 0, \quad n > 0 \tag{11}$$

Similarly,

$$\int_0^L dx\sin\left(\frac{2\pi n}{L}x\right) = 0, \quad n > 0 \tag{12}$$

Thus,

$$\int_0^L dx\, f(x) = a_0\int_0^L dx + \sum_{n=1}^{\infty} a_n\int_0^L dx\cos\left(\frac{2\pi n}{L}x\right) + \sum_{n=1}^{\infty} b_n\int_0^L dx\sin\left(\frac{2\pi n}{L}x\right) \tag{13}$$

$$= a_0 L \tag{14}$$

in agreement with Eq. (8).

For a_n we can use the cosine sum formula to write

$$\int_0^L dx\cos\left(\frac{2\pi m}{L}x\right)\cos\left(\frac{2\pi n}{L}x\right) = \int_0^L dx\left[\frac{1}{2}\cos\left(\frac{n+m}{L}2\pi x\right) + \frac{1}{2}\cos\left(\frac{n-m}{L}2\pi x\right)\right] \tag{15}$$

Now again we have that these integrals all vanish over an integer number of periods of the cosine curve. The only way this wouldn't vanish is if $n - m = 0$. So we have for $n > 0$

$$\int_0^L dx\cos\left(\frac{2\pi m}{L}x\right)\cos\left(\frac{2\pi n}{L}x\right) = \frac{1}{2}\delta_{mn}\int_0^L dx = \frac{L}{2}\delta_{mn} \tag{16}$$

where δ_{mn} is the **Kronecker $\boldsymbol{\delta}$-function**

$$\delta_{mn} = \begin{cases} 0, & m \neq n \\ 1 & m = n \end{cases} \tag{17}$$

Similarly,

$$\int_0^L dx\cos\left(\frac{2\pi m}{L}x\right)\sin\left(\frac{2\pi n}{L}x\right) = 0 \tag{18}$$

$$\int_0^L dx\sin\left(\frac{2\pi m}{L}x\right)\sin\left(\frac{2\pi n}{L}x\right) = \frac{L}{2}\delta_{mn} \tag{19}$$

Thus, for $n > 0$

$$\int_0^L dx\, f(x)\cos\left(\frac{2\pi n}{L}x\right) \tag{20}$$

$$= \int_0^L dx\left[a_0 + \sum_{m=1}^{\infty} a_m\cos\left(\frac{2\pi m}{L}x\right) + \sum_{m=1}^{\infty} b_m\sin\left(\frac{2\pi m}{L}x\right)\right]\cos\left(\frac{2\pi n}{L}x\right) \tag{21}$$

$$= \frac{L}{2}\sum_{m=1}^{\infty} a_m\delta_{mn} \tag{22}$$

$$= \frac{L}{2}a_n \tag{23}$$

as in Eq. (9). In the same way you can check the formula for b_n.

We use

- Fourier cosine series for functions which are **even** on the interval $(f(x) = f(L - x))$
- Fourier sine series for functions which are **odd** on the interval $(f(x) = -f(L - x))$
- For functions that are neither even nor odd on the interval, we need both sines and cosines

3 Example

Find the Fourier series for the sawtooth function:

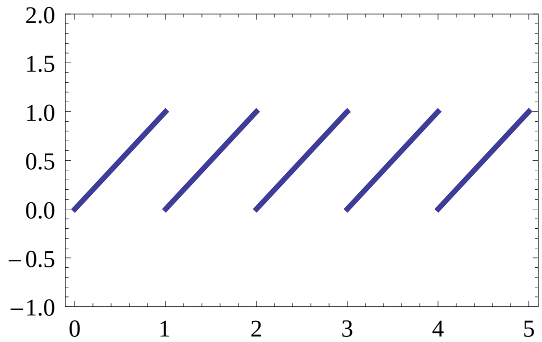


Figure 1. Sawtooth function

This function is clearly periodic. It is equal to $f(x) = x$ on the interval $0 < x \leqslant 1$. Thus we can compute the Fourier series with $L = 1$. We get

$$a_0 = \frac{1}{L}\int_0^L f(x) = \int_0^1 dx\,x = \frac{1}{2} \tag{24}$$

Next,

$$a_n = \frac{2}{L}\int_0^L dx\,f(x)\cos\left(\frac{2\pi n}{L}x\right) = 2\int_0^1 dx\,x\cos(2\pi n x) \tag{25}$$

This can be done by integration by parts

$$a_n = 2\frac{x}{2\pi n}\sin(2\pi n x)\Big|_0^L - 2\int_0^1 dx\sin(2\pi n x) = 0 \tag{26}$$

Finally,

$$b_n = 2\int_0^1 dx\,x\sin(2\pi n x) \tag{27}$$

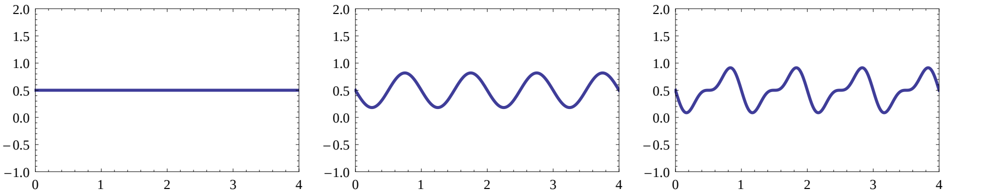
$$= -2\frac{x}{2\pi n}\cos(2\pi n x)\Big|_0^L + 2\int_0^1 dx\cos(2\pi n x) \tag{28}$$

$$= -\frac{1}{\pi n} \tag{29}$$

Thus we have

$$\boxed{f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} -\frac{1}{\pi n}\sin\frac{2\pi n x}{L}} \tag{30}$$

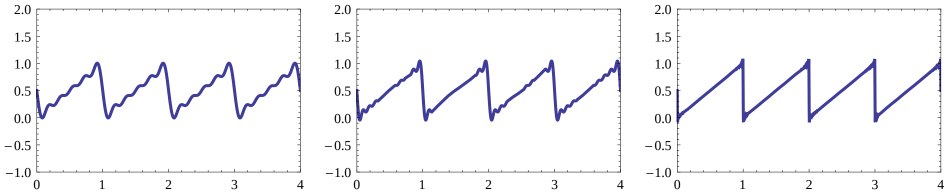
Let's look at how well the series approximates the function when including various terms. Taking 0, 1 and 2 terms in the sum gives



1 mode $f=\frac{1}{2}$ | 2 modes $f=\frac{1}{2}-\frac{1}{\pi}\sin(2\pi x)$ | 3 modes $f=\frac{1}{2}-\frac{1}{\pi}\sin(2\pi x)-\frac{1}{2\pi}\sin(4\pi x)$

Figure 2. Approximations to the sawtooth function

Already at 3 modes, it's looking reasonable. For 5, 10 and 100 modes we find



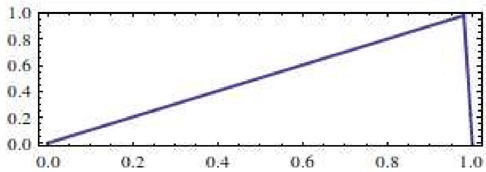
5 mode | 10 modes | 100 modes

Figure 3. More approximations to the sawtooth function

For 10 modes we find excellent agreement.

4 Plucking a string

Let's apply the Fourier decomposition we worked out to plucking a string. Suppose we pluck a string by pulling up one end:



What happens to the string? To find out, let us do a Fourier decomposition of the x-dependence of the pluck. We start by writing

$$A(x,t)=\sum_{n=0}^{\infty}\left[a_n\cos\left(\frac{2n\pi}{L}x\right)\cos(\omega_n t)+b_n\sin\left(\frac{2n\pi}{L}x\right)\cos(\omega_n t)\right],\quad \omega_n=\frac{2n\pi}{L}v \tag{31}$$

Here v is the speed of sound in the string. For a given wavenumber, $k_n=\frac{2n\pi}{L}$, we know that $\omega_n=k_n v$ to satisfy the wave equation. We could also have included components with $\sin(\omega_n t)$; however since the string starts off at rest (so that $\partial_t A(x,\,t)=0$), then the coefficients of the $\sin(\omega_n t)$ functions must all vanish.

At time $t=0$, the amplitude is

$$A(x,0)=\sum_{n=0}^{\infty}\left[a_n\cos\left(\frac{2n\pi}{L}x\right)+b_n\sin\left(\frac{2n\pi}{L}x\right)\right] \tag{32}$$

This is just the Fourier decomposition of the function described by our pluck shape. If we approximate the pluck as the sawtooth function from the previous section, then we already know that

$$a_n=0,\quad b_n=-\frac{1}{\pi n} \tag{33}$$

So that, setting $L=1$

$$A(x,t)=\sum_{n=1}^{\infty}-\frac{1}{\pi n}\sin(2\pi x)\cos(2\pi v t) \tag{34}$$

This gives the motion of the string for all time.

The relative amplitudes of each mode are

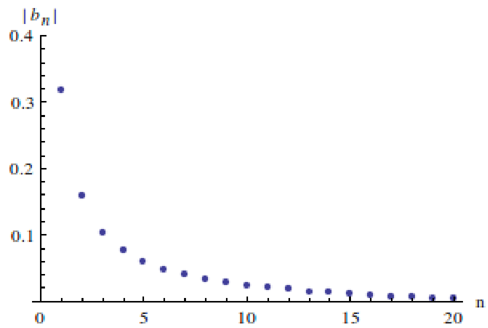


Figure 4. Amplitudes of the relative harmonics of a string plucked with a sawtooth plucking.

The $n=1$ mode is the largest. This the **fundamental** frequency of the string. Thus the sound that comes out of the string will be mostly this frequency: $\omega_1=\frac{2\pi}{L}v$. The modes with $n>1$ are the **harmonics**. Harmonics have frequencies which are integer multiples of the fundamental.

5 Exponentials

Fourier series decompositions are even easier with complex numbers. There we can replace the sines and cosines by exponentials. The series is

$$\boxed{f(x)=\sum_{n=-\infty}^{\infty} c_n e^{inx\frac{2\pi}{L}}} \tag{35}$$

where

$$c_n=\frac{1}{L}\int_0^L dx\, f(x)e^{-inx\frac{2\pi}{L}} \tag{36}$$

To check this, we substitute in

$$\int_0^L dx\, f(x)e^{-inx\frac{2\pi}{L}}=\sum_{m=-\infty}^{\infty} c_m\int_0^L dx\, e^{imx\frac{2\pi}{L}}e^{-inx\frac{2\pi}{L}}=\sum_{m=\infty}^{\infty} c_m\int_0^L dx\, e^{i(m-n)x\frac{2\pi}{L}} \tag{37}$$

If $n\neq m$ then,

$$\int_0^L dx\, e^{i(m-n)x\frac{2\pi}{L}}=\frac{L}{2\pi}\frac{1}{m-n}e^{i(m-n)\frac{2\pi}{L}x}\Big|_0^L \tag{38}$$

$$=\frac{L}{2\pi}\frac{1}{m-n}\left[e^{i2\pi(m-n)}-1\right]=0 \tag{39}$$

If $m=n$, then the integral is just

$$\int_0^L dx=L \tag{40}$$

Thus,

$$\int_0^L dx\, e^{i(m-n)x\frac{2\pi}{L}}=L\delta_{mn} \tag{41}$$

and so

$$\int_0^L dx\, f(x)e^{-inx\frac{2\pi}{L}}=Lc_n \tag{42}$$

If $f(x)$ is real, then

$$f(x)=\sum_{n=-\infty}^{\infty}\mathrm{Re}(c_n+c_{-n})\cos\left(\frac{2\pi n x}{L}\right)+\mathrm{Im}(c_{-n}+c_n)\sin\left(\frac{2\pi n x}{L}\right) \tag{43}$$

So $a_n=\mathrm{Re}(c_n+c_{-n})$ and $b_n=\mathrm{Im}(c_{-n}+c_n)$. Thus, the exponential series contains all the information in both the sine and cosine series in an efficient form.

6 Orthogonal functions (optional)

In verifying Fourier's theorem, we found the relevant integral equations

$$\frac{1}{L}\int_0^L dx\, e^{i\,(m-n)\,x\frac{2\pi}{L}}=\delta_{mn} \tag{44}$$

$$\frac{1}{L}\int_0^L dx\cos\left(\frac{2\pi m}{L}x\right)\sin\left(\frac{2\pi n}{L}x\right)=0 \tag{45}$$

$$\frac{1}{L}\int_0^L dx\cos\left(\frac{2\pi m}{L}x\right)\cos\left(\frac{2\pi n}{L}x\right)=\frac{1}{2}\delta_{nm} \tag{46}$$

$$\frac{1}{L}\int_0^L dx\sin\left(\frac{2\pi m}{L}x\right)\sin\left(\frac{2\pi n}{L}x\right)=\frac{1}{2}\delta_{nm} \tag{47}$$

These are examples of orthogonal functions. The integral is a type of **inner product**. The dot-product among vectors is another example of an inner product. We can write the inner product in various ways

$$\langle v\,|w\rangle\equiv\vec{v}\cdot\vec{w}=\sum_i v_i w_i \tag{48}$$

The integral inner product is a generalization of this from vectors of numbers to functions.

We can define the inner product of two functions as

$$\langle f\,|g\rangle=\frac{1}{2\pi}\int_0^{2\pi} dx\, f^\star(x)g(x) \tag{49}$$

where $f^\star(x)$ is the complex conjugate of $f(x)$. For example,

$$\langle e^{imx}|e^{inx}\rangle=\delta_{mn} \tag{50}$$

This is the analog of

$$\langle x_i\,|x_j\rangle=\delta_{ij} \tag{51}$$

where $|x_i\rangle=(0,\cdots,0,1,0,0)$ with the 1 in the i^{th} component. That is the $|x_i\rangle$ are the unit vectors. When a set of functions satisfy

$$\langle f_i|f_j\rangle=\delta_{ij} \tag{52}$$

we say that they are **orthonormal**. The **ortho** part means they are **orthogonal**: $\langle f_i|f_j\rangle=0$ for $i\neq j$. The **normal** part means they are normalized, $\langle f_i|f_j\rangle=1$ for $i=j$.

If any function can be written as a linear combination of functions f_i we say that the set $\{f_i\}$ is **complete**. Then

$$f(x)=\sum_i a_i f_i(x) \tag{53}$$

We can extract a_i via

$$\langle f(x)|f_i\rangle=\sum_j a_j\langle f_j|f_i\rangle=\sum_j a_j\delta_{ij}=a_i \tag{54}$$

This is exactly what we did with the Fourier decomposition above. It is also what we do with vectors

$$\vec{v} = \sum c_i \vec{x}_i \tag{55}$$

Then $c_i = \langle v|x_i\rangle$ which is just the i^{th} component of $\vec{v}$.

We will see various sets of orthonormal function bases with different inner products come up in physics. Other examples are:

Bessel functions: $\mathcal{J}_n(x)$. These functions are the solutions to the differential equation

$$x^2 f''(x) + x f'(x) + (x^2 - n^2) f(x) = 0 \tag{56}$$

They satisfy the orthonormality condition

$$\langle \mathcal{J}_n|\mathcal{J}_n\rangle = \int_0^1 dx\, x\, \mathcal{J}_n(x) \mathcal{J}_m(x) = \delta_{nm} \tag{57}$$

Bessel functions come up in 2 dimensional problems. You will start to see them all over the place in physics.

Legendre polynomials P_n. These satisfy $P_0(x) = 1, \quad P_1(x) = x$ and

$$(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x) \tag{58}$$

Their inner product is

$$\langle P_n|P_m\rangle = \int_0^1 dx\, P_n(x) P_m(x) = \frac{2}{2n+1} \delta_{nm} \tag{59}$$

Legendre polynomials come up in problems with spherical symmetry. You will study them to death in quantum mechanics.

Hermite polynomials

$$H_n(x) = (-1)^n e^{-\frac{x^2}{2}} \frac{d}{dx^2} e^{-\frac{x^2}{2}} \tag{60}$$

So $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$ and so on. These satisfy

$$\langle H_n|H_m\rangle = \int_0^1 dx\, e^{-\frac{x^2}{2}} H_n(x) H_m(x) = \delta_{nm} \tag{61}$$

Hermite polynomials play a critical role in the quantum harmonic oscillator.